

GENERALISED JANTZEN FILTRATION OF LIE SUPERALGEBRAS II: THE EXCEPTIONAL CASES

YUCAI SU AND R.B. ZHANG

ABSTRACT. Let \mathfrak{g} be an exceptional Lie superalgebra, and let \mathfrak{p} be the maximal parabolic subalgebra which contains the distinguished Borel subalgebra and has a purely even Levi subalgebra. For any parabolic Verma module in the parabolic category $\mathcal{O}^{\mathfrak{p}}$, it is shown that the Jantzen filtration is the unique Loewy filtration, and the decomposition numbers of the layers of the filtration are determined by the coefficients of inverse Kazhdan-Lusztig polynomials. An explicit description of the submodule lattices of the parabolic Verma modules is given, and formulae for characters and dimensions of the finite dimensional simple modules are obtained.

CONTENTS

1. Introduction	2
2. Exceptional Lie superalgebras	3
3. Generalised Jantzen filtration	6
3.1. Parabolic category $\mathcal{O}^{\mathfrak{p}}$	7
3.2. Generalised Jantzen filtration	8
4. Classification of atypical weights	10
4.1. The up and down moves	10
4.2. Description of P_{aty}^1 and P_{aty}^2	11
4.3. A classification of integral atypical weights	13
5. Structure of parabolic Verma modules	14
5.1. Primitive weight graphs	14
5.2. The \mathfrak{g}_0 -highest weights in parabolic Verma modules	15
5.3. Structure theorem for parabolic Verma modules	17
6. Proofs of main theorems on Jantzen filtration	21
6.1. Rigidity of parabolic Verma modules	21
6.2. Proof of Theorem 3.3	22
6.3. Computation of \mathfrak{u}^- -homology groups	24
6.4. Proof of Theorem 3.5	26
7. Characters, dimensions and cohomology groups of finite dimensional simple modules	27
7.1. Character and dimension formulae for simple modules	27
7.2. First and second cohomology groups	28
8. Comments on Jantzen filtration for orthosymplectic Lie superalgebras	28
References	29

1. INTRODUCTION

In this paper we continue the investigation started in [32] on Jantzen type filtration for classical Lie superalgebras [20, 25].

The Jantzen filtration was first introduced by Jantzen [18, 19] for Verma modules over semi-simple complex Lie algebras in the BGG category \mathcal{O} . It soon became clear that its properties were deeply rooted in Kazhdan-Lusztig theory. Much work was devoted to studying Jantzen filtration (see, e.g., [15, 13, 5, 17] and references therein) in the 80s, culminating at the celebrated proof of the Jantzen conjectures [3, 4] by generalising geometric techniques used in the proof [3, 8] of the Kazhdan-Lusztig conjecture [24]. The circle of ideas surrounding Jantzen filtration and its generalisations [1, 2] led to the development [27, 14, 28] (see [16] for further references) of deformation techniques capable of reaching deeper properties of category \mathcal{O} which are otherwise difficult to see.

Let \mathfrak{g} be a simple basic classical Lie superalgebra [20, 25] or $\mathfrak{gl}_{m|n}$ defined over the field \mathbb{C} of complex numbers. We fix once for all the chain of subalgebras $\mathfrak{h} \subset \mathfrak{b} \subset \mathfrak{p} \subset \mathfrak{g}$, where \mathfrak{h} is a Cartan subalgebra, \mathfrak{b} is the distinguished Borel subalgebra in the sense of [20], and $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{u}$ is the maximal parabolic subalgebra with the nilradical \mathfrak{u} and purely even Levi subalgebra \mathfrak{l} . Denote by $V(\lambda)$ the parabolic Verma module with highest weight λ in the parabolic category $\mathcal{O}^{\mathfrak{p}}$ of \mathbb{Z}_2 -graded \mathfrak{g} -modules. In [32], we introduced a Jantzen type filtration

$$V(\lambda) = V^0(\lambda) \supset V^1(\lambda) \supset V^2(\lambda) \supset \cdots \supset V^\ell(\lambda) \supset 0, \quad \text{where } V^\ell(\lambda) \neq 0,$$

for each parabolic Verma module $V(\lambda)$ by studying a deformation of the parabolic category $\mathcal{O}^{\mathfrak{p}}$. When \mathfrak{g} is a type I Lie superalgebra (consisting of $\mathfrak{gl}_{m|n}$, $A(k|l)$ and $\mathfrak{osp}_{2|2n}$), we proved that

- (i) the Jantzen filtration is the unique Loewy filtration of the parabolic Verma module $V(\lambda)$;
- (ii) the decomposition numbers of the consecutive quotients of the Jantzen filtration are described by the coefficients of the inverse Kazhdan-Lusztig polynomials; and
- (iii) the length ℓ of the Jantzen filtration of $V(\lambda)$ is equal to the degree of atypicality of the highest weight λ .

Recall that a filtration of a module is Loewy if its consecutive quotients are all semi-simple, and it has the smallest length among all such filtrations. The generalised Kazhdan-Lusztig polynomials are defined in terms of \mathfrak{u} -cohomology groups $H^i(\mathfrak{u}, L(\lambda))$. The degree of atypicality of λ is the number of isotropic odd positive roots γ_i such that $(\lambda + \rho, \gamma_i) = 0$ and $(\gamma_i, \gamma_j) = 0$.

In the present paper we show that if \mathfrak{g} is any of the exceptional Lie superalgebras $D(2, 1; a)$, F_4 and G_3 , the Jantzen filtration of a parabolic Verma module $V(\lambda)$ in $\mathcal{O}^{\mathfrak{p}}$ satisfies properties (i) and (ii). In particular, parabolic Verma modules are rigid. The precise statements of these results are given in Theorems 3.2, 3.3 and 3.5.

Property (iii) no longer holds for any exceptional Lie superalgebra. For any $\lambda \in \mathfrak{h}^*$ which is integral dominant, $V(\lambda)$ is not simple regardless of whether λ is typical (i.e., with the degree of atypicality being 0) or not. Thus property (iii) can not hold for such highest weights. Another important fact is that category $\mathcal{O}^{\mathfrak{p}}$ is not the category of finite dimensional \mathbb{Z}_2 -graded \mathfrak{g} -modules, in sharp contrast to the type I case.

Let us list the other main results of this paper.

(1). We explicitly describe in Theorem 5.6 the submodule lattice of any parabolic Verma module $V(\lambda)$ in \mathcal{O}^p for all exceptional Lie superalgebras. This is a significant result in its own right. Here it enables us to prove the main results on Jantzen filtration. The method used to prove Theorem 5.6 is a generalisation of that in [33], where we worked out the structure of the parabolic Verma modules for $\mathfrak{osp}_{k|2}$. It involves finding the primitive vectors in $V(\lambda)$.

(2). We compute the \mathfrak{u} -cohomology groups with coefficients in any simple \mathfrak{g} -module with the help of Theorem 5.6. The result is given in Theorem 6.3 (where we actually give the \mathfrak{u}^- -homology groups, but see Remark 3.4). This result provides crucial information needed for proving property (ii) of the Jantzen filtration of $V(\lambda)$.

(3). We determine the character and dimension of any finite dimensional simple \mathfrak{g} -module by using Theorem 5.6, and give explicit formulae for them in Theorem 7.1. Note that in the case of $D(2, 1; a)$, characters and dimensions of finite dimensional simple modules were obtained in [34].

(4). We obtain in Theorem 7.2 the first and second \mathfrak{g} -cohomology groups with coefficients in the finite dimensional simple and Kac modules. Analogous results were obtained for the type I Lie superalgebras in [30] and for $\mathfrak{osp}_{k|2}$ in [33].

Finally we comment briefly on the method used in this paper. Recall that essential use was made of knowledge on the generalised Kazhdan-Lusztig polynomials [26, 6, 35] and super duality [12, 11, 9, 7, 10] when establishing the above listed properties of the Jantzen filtration for the type I superalgebras in [32]. For the exceptional Lie superalgebras, no such results are available. Thus in this paper, we will take a “bottom-up” approach instead by understanding first the submodule lattice of parabolic Verma modules then deducing the properties of the Jantzen filtration.

2. EXCEPTIONAL LIE SUPERALGEBRAS

In this section we present some preliminary material which will be needed later. Throughout the paper, we work over the field \mathbb{C} of complex numbers.

Let \mathfrak{g} be one of the exceptional Lie superalgebras, that is, \mathfrak{g} is either $D(2, 1; a)$ ($a \neq 0, -1$), F_4 or G_3 . Relative to the distinguished Borel subalgebra \mathfrak{b} , the Cartan matrix of \mathfrak{g} is given by

$$A = \begin{pmatrix} 0 & 1 & a \\ -1 & 2 & 0 \\ -1 & 0 & 2 \end{pmatrix} \begin{matrix} 0 \\ 1 \\ 2 \end{matrix}, \quad \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -2 & 0 \\ 0 & -1 & 2 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \begin{matrix} 1 \\ 2 \\ 3 \\ 0 \end{matrix} \quad \text{or} \quad \begin{pmatrix} 0 & 1 & 0 \\ -1 & 2 & -3 \\ -1 & 0 & 2 \end{pmatrix} \begin{matrix} 0 \\ 1 \\ 2 \end{matrix}$$

(where the numbers beside the matrix are the row indices) and Dynkin diagram by

$$\begin{array}{c} \alpha_0 \\ \circledast \end{array} \begin{array}{c} \nearrow \\ \searrow \end{array} \begin{array}{c} \alpha_1 \\ \circ \end{array} \quad , \quad \begin{array}{c} \alpha_1 \\ \circ \end{array} \begin{array}{c} \alpha_2 \\ \circ \end{array} \begin{array}{c} \alpha_3 \\ \circ \end{array} \begin{array}{c} \alpha_0 \\ \circledast \end{array} \quad \text{or} \quad \begin{array}{c} \alpha_0 \\ \circledast \end{array} \begin{array}{c} \alpha_1 \\ \circ \end{array} \begin{array}{c} \alpha_2 \\ \circ \end{array} \quad .$$

We realize the root system of \mathfrak{g} in the vector space $E = \bigoplus_{i=0}^{I_1} \mathbb{C}\varepsilon_i$, ($I_1 = 2$ if \mathfrak{g} is $D(2, 1; a)$, and 3 otherwise) equipped with a non-degenerate symmetric bilinear form

such that $\{\delta := \varepsilon_0, \varepsilon_i \mid i = 1, \dots, I_1\}$ is an orthogonal basis satisfying

$$\begin{aligned} D(2, 1; a) : (\delta, \delta) &= -(1+a), (\varepsilon_1, \varepsilon_1) = 1, (\varepsilon_2, \varepsilon_2) = a, \\ F_4 : (\delta, \delta) &= -6, (\varepsilon_i, \varepsilon_j) = 2\delta_{ij}, i, j = 1, 2, 3, \\ G_3 : (\delta, \delta) &= -2, (\varepsilon_i, \varepsilon_j) = \delta_{ij}, i, j = 1, 2, 3. \end{aligned} \quad (2.1)$$

Denote by Π the set of the simple roots, by Δ_0^+ (resp. Δ_1^+) the set of even (resp. odd) positive roots. For convenience, we write Δ_1^+ as the disjoint union $\Delta_1^+ = \Delta_1^\dagger \cup \Delta_1^\pm$. Then we have

$$\begin{aligned} D(2, 1; a) : \quad \Pi &= \{\alpha_0 = \delta - \varepsilon_1 - \varepsilon_2, \alpha_1 = 2\varepsilon_1, \alpha_2 = 2\varepsilon_2\}, \\ \Delta_0^+ &= \{2\varepsilon_1, 2\varepsilon_2, 2\delta\}, \\ \Delta_1^\dagger &= \{\delta + \varepsilon_1 \pm \varepsilon_2\}, \quad \Delta_1^\pm = \{\delta - \varepsilon_1 \pm \varepsilon_2\}; \end{aligned} \quad (2.2)$$

$$\begin{aligned} F_4 : \quad \Pi &= \{\alpha_1 = \varepsilon_1 - \varepsilon_2, \alpha_2 = \varepsilon_2 - \varepsilon_3, \alpha_3 = \varepsilon_3, \alpha_0 = \frac{1}{2}(\delta - \varepsilon_1 - \varepsilon_2 - \varepsilon_3)\}, \\ \Delta_0^+ &= \{\delta, \varepsilon_i, \varepsilon_i \pm \varepsilon_j \mid i < j = 1, 2, 3\}, \\ \Delta_1^\dagger &= \left\{ \frac{1}{2}(\delta + \varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3) \right\}, \quad \Delta_1^\pm = \left\{ \frac{1}{2}(\delta - \varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3) \right\}; \end{aligned} \quad (2.3)$$

$$\begin{aligned} G_3 : \quad \Pi &= \{\alpha_0 = \delta - \varepsilon_1 + \varepsilon_3, \alpha_1 = \varepsilon_1 - \varepsilon_2, \alpha_3 = 2\varepsilon_2 - \varepsilon_1 - \varepsilon_3\}, \\ \Delta_0^+ &= \{2\delta, \varepsilon_1 - \varepsilon_2, \varepsilon_1 - \varepsilon_3, \varepsilon_2 - \varepsilon_3, \\ &\quad 2\varepsilon_1 - \varepsilon_2 - \varepsilon_3, 2\varepsilon_2 - \varepsilon_1 - \varepsilon_3, \varepsilon_1 + \varepsilon_2 - 2\varepsilon_3\}, \\ \Delta_1^\dagger &= \{\delta + \varepsilon_1 - \varepsilon_2, \delta + \varepsilon_1 - \varepsilon_3, \delta + \varepsilon_2 - \varepsilon_3\}, \\ \Delta_1^\pm &= \{\delta, \delta - \varepsilon_1 + \varepsilon_2, \delta - \varepsilon_1 + \varepsilon_3, \delta - \varepsilon_2 + \varepsilon_3\}. \end{aligned} \quad (2.4)$$

Let $\Delta^+ = \Delta_0^+ \cup \Delta_1^+$ and $\Delta = \Delta^+ \cup (-\Delta^+)$. We choose the Chevalley generators $e_i = e_{\alpha_i}, f_i = f_{\alpha_i}, h_i, i = 0, \dots, I_2$, where $I_2 = 3$ if $\mathfrak{g} = F_4$ and $I_2 = 2$ otherwise. They satisfy the commutation relations

$$\begin{aligned} [e_i, f_j] &= \delta_{ij} h_i, \\ [h_i, e_j] &= \frac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)} e_j, \quad [h_i, f_j] = -\frac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)} f_j, \quad i \neq 0, \\ [h_0, e_j] &= \frac{1}{2}(\alpha_0, \alpha_j) e_j, \quad [h_0, f_j] = -\frac{1}{2}(\alpha_0, \alpha_j) f_j. \end{aligned} \quad (2.5)$$

We define the composite root vectors e_α, f_α with $\alpha \in \Delta^+ \setminus \Pi$ as follows: Let $i \in \{0, \dots, I_2\}$ be the smallest such that $\alpha = \beta + \alpha_i$ for some $\beta \in \Delta^+$, and set $e_\alpha = [e_\beta, e_{\alpha_i}], f_\alpha = [f_{\alpha_i}, f_\beta]$. For instance, in the case of $D(2, 1; a)$,

$$\begin{aligned} e_{\delta+\varepsilon_1-\varepsilon_2} &:= [e_1, e_0], \quad f_{\delta+\varepsilon_1-\varepsilon_2} := [f_0, f_1], \\ e_{\delta-\varepsilon_1+\varepsilon_2} &:= [e_2, e_0], \quad f_{\delta-\varepsilon_1+\varepsilon_2} := [f_0, f_2], \\ e_{\delta+\varepsilon_1+\varepsilon_2} &:= [e_{\delta-\varepsilon_1+\varepsilon_2}, e_1] = [e_{\delta+\varepsilon_1-\varepsilon_2}, e_2] = [[e_0, e_1], e_2], \\ f_{\delta+\varepsilon_1+\varepsilon_2} &:= [f_1, f_{\delta-\varepsilon_1+\varepsilon_2}], \\ e_{2\delta} &:= [e_{\delta+\varepsilon_1+\varepsilon_2}, e_0] = -[e_{\delta+\varepsilon_1-\varepsilon_2}, e_{\delta-\varepsilon_1+\varepsilon_2}], \\ f_{2\delta} &:= [f_0, f_{\delta+\varepsilon_1+\varepsilon_2}]. \end{aligned} \quad (2.6)$$

Then we have

$$\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+, \quad \text{where } \mathfrak{h} = \bigoplus_{i=0}^{I_2} \mathbb{C}h_i, \quad \mathfrak{n}^\pm = \bigoplus_{\alpha \in \Delta^\pm} \mathfrak{g}^{\pm\alpha},$$

with $\mathfrak{g}^\alpha = \mathbb{C}e_\alpha$, $\mathfrak{g}^{-\alpha} = \mathbb{C}f_\alpha$. There is an anti-involution τ of the universal enveloping algebra $U(\mathfrak{g})$ such that

$$\tau(\mathfrak{g}^\alpha) = \mathfrak{g}^{-\alpha}, \quad \tau(h) = h, \quad \alpha \in \Delta, \quad h \in \mathfrak{h}. \quad (2.7)$$

Here by τ being an anti-involution we mean that $\tau^2 = 1$ and $\tau(xy) = \tau(y)\tau(x)$ for all $x, y \in U(\mathfrak{g})$. Note that there is no sign factor on the right hand side of the second formula when both x and y are odd. One may introduce a sign factor to obtain a graded version of τ , which however will not play any role in this paper.

Remark 2.1. It follows from the definition of e_α that $\tau(e_\alpha)$ is a nonzero scalar multiple of f_α . This scalar factor will not play any role in later computations.

Observe that \mathfrak{g} admits a \mathbb{Z} -gradation compatible with its \mathbb{Z}_2 -gradation

$$\mathfrak{g} = \bigoplus_{k=-2}^2 \mathfrak{g}_k, \quad \text{such that } \deg e_i = -\deg f_i = \delta_{i,0}, \quad \deg h_i = 0, \quad (2.8)$$

$$\mathfrak{g}_2 = \mathbb{C}e_\varphi, \quad \text{where } \varphi = \delta \text{ if } \mathfrak{g} = F_4, \text{ and } \varphi = 2\delta \text{ otherwise.} \quad (2.9)$$

Here \mathfrak{g}_0 is a reductive Lie algebra with positive root system $\Delta_0^+ = \Delta_0^+ \setminus \{\varphi\}$:

$$\mathfrak{g}_0 \cong \begin{cases} \mathbb{C}h_0 \oplus sl_2 \oplus sl_2 & \text{if } \mathfrak{g} = D(2, 1; a), \\ \mathbb{C}h_0 \oplus so(7) & \text{if } \mathfrak{g} = F_4, \\ \mathbb{C}h_0 \oplus G_2 & \text{if } \mathfrak{g} = G_3. \end{cases}$$

We choose $h_\varphi \in \mathfrak{h}$ to be the unique element such that

$$[h_\varphi, e_\alpha] = \frac{2(\varphi, \alpha)}{(\varphi, \varphi)} e_\alpha \quad \text{for all } \alpha \in \Delta^+. \quad (2.10)$$

For any $\lambda \in \mathfrak{h}^*$ and $h = \sum_{j=0}^{I_2} c_j h_j$ in \mathfrak{h} , we have

$$\lambda(h) = \frac{c_0}{2}(\alpha_0, \lambda) + \sum_{j=1}^{I_2} c_j \frac{2(\alpha_j, \lambda)}{(\alpha_j, \alpha_j)}. \quad (2.11)$$

Write any $\lambda \in \mathfrak{h}^*$ in terms of the $\delta\varepsilon$ -basis as

$$\lambda = \lambda_0 \delta + \sum_{i=1}^{I_1} \lambda_i \varepsilon_i = (\lambda_0 \mid \lambda_1, \lambda_2, \dots, \lambda_{I_1}),$$

and say that λ_i is the i -th coordinate of λ ($i = 0, \dots, I_1$). However, we should bear in mind that in the case of G_3 , the roots span a proper subspace of E , and by (2.11) the following elements have the same projections onto the subspace

$$(\lambda_0 \mid \lambda_1, \lambda_2, \lambda_3), \quad (\lambda_0 \mid \lambda_1 + x, \lambda_2 + x, \lambda_3 + x) \quad \text{for any } x \in \mathbb{C} \text{ if } \mathfrak{g} = G_3. \quad (2.12)$$

Let ρ_0 (resp. ρ_1) be half of the sum of the even (resp. odd) positive roots, and let $\rho = \rho_0 - \rho_1$. Then we have the following table.

\mathfrak{g}	ρ_0	ρ_1	ρ
$D(2, 1; a)$	$(1 \mid 1, 1)$	$(2 \mid 0, 0)$	$(-1 \mid 1, 1)$
F_4	$\left(\frac{1}{2} \mid \frac{5}{2}, \frac{3}{2}, \frac{1}{2}\right)$	$(2 \mid 0, 0, 0)$	$\left(-\frac{3}{2} \mid \frac{5}{2}, \frac{3}{2}, \frac{1}{2}\right)$
G_3	$(1 \mid 2, 1, -3)$	$\left(\frac{7}{2} \mid 0, 0, 0\right)$	$\left(-\frac{5}{2} \mid 2, 1, -3\right)$

(2.13)

Given any $\lambda \in \mathfrak{h}^*$, we let $\lambda^\rho = \lambda + \rho$ with coordinates denoted by λ_i^ρ , namely

$$\lambda^\rho = \lambda + \rho = (\lambda_0^\rho \mid \lambda_1^\rho, \dots, \lambda_{I_1}^\rho). \quad (2.14)$$

Let σ_i be the reflection on E which acts by changing the sign of the i -th coordinate of any element λ . In the cases of F_4 and G_3 , the symmetry group Sym_3 of rank 3 acts on E by permuting the coordinates $\lambda_1, \lambda_2, \lambda_3$ of any element λ . Denote $\bar{\sigma} = \sigma_1\sigma_2\sigma_3$, which changes the signs of the coordinates $\lambda_1, \lambda_2, \lambda_3$ simultaneously.

Denote by W_0 and W respectively the Weyl groups of \mathfrak{g}_0 and \mathfrak{g} . Then $W = W_0 \times \langle \sigma_0 \rangle$ and

$$W_0 = \begin{cases} \langle \sigma_1, \sigma_2 \rangle & \text{if } \mathfrak{g} = D(2, 1; a), \\ \text{Sym}_3 \ltimes \langle \sigma_1, \sigma_2, \sigma_3 \rangle & \text{if } \mathfrak{g} = F_4, \\ \text{Sym}_3 \times \langle \bar{\sigma} \rangle & \text{if } \mathfrak{g} = G_3. \end{cases} \quad (2.15)$$

Here $\bar{\sigma}$ denotes its restriction on the span of the roots of G_3 . Observe from (2.12) that $\bar{\sigma}$ is in fact of length 2 in the Weyl group.

As usual, we define the *dot action* of W on \mathfrak{h}^* by $w \cdot \lambda = w(\lambda + \rho) - \rho$ for any $w \in W$ and $\lambda \in \mathfrak{h}^*$. Let $\lambda^{\sigma_0} := \sigma_0 \cdot \lambda$. Then

$$\lambda^{\sigma_0} := (c - \lambda_0 \mid \lambda_1, \dots, \lambda_{I_1}), \quad (2.16)$$

with $c = 2, 3$ and 5 for $\mathfrak{g} = D(2, 1; a), F_4$ and G_3 respectively.

We have the usual partial order “ \preceq ” on \mathfrak{h}^* given by

$$\lambda \preceq \mu \iff \mu - \lambda \in \mathbb{Z}_+\Pi = \left\{ \sum_{i=0}^{I_1} m_i \alpha_i \mid m_i \in \mathbb{Z}_+ \right\}. \quad (2.17)$$

Let $\Gamma = \mathbb{Z}\Delta^+$ be the root lattice endowed with the lexicographical order, namely, for $x = \sum_{i=0}^{I_1} x_i \alpha_i$ and $y = \sum_{i=0}^{I_1} y_i \alpha_i$,

$$x < y \iff \text{for the first } i \text{ with } x_i \neq y_i, \text{ we have } x_i < y_i. \quad (2.18)$$

3. GENERALISED JANTZEN FILTRATION

Let \mathfrak{g} be an exceptional Lie superalgebra, and let $\mathfrak{p} = \mathfrak{l} + \mathfrak{u}$ be the maximal parabolic subalgebra with the purely even Levi subalgebra $\mathfrak{l} = \mathfrak{g}_0$ and nilradical $\mathfrak{u} = \mathfrak{g}_{+1} + \mathfrak{g}_{+2}$. Then $\mathfrak{p} \supset \mathfrak{b} \supset \mathfrak{h}$, where \mathfrak{b} is the distinguished Borel subalgebra. We shall refer to \mathfrak{p} as the *distinguished maximal parabolic subalgebra* for convenience. Let $\mathfrak{u}^- = \mathfrak{g}_{-1} \oplus \mathfrak{g}_{-2}$, then $\mathfrak{g} = \mathfrak{u}^- \oplus \mathfrak{p}$. The notation will be in force throughout the paper.

3.1. Parabolic category \mathcal{O}^p . We consider the parabolic category \mathcal{O}^p of \mathbb{Z}_2 -graded \mathfrak{g} -modules, which are finitely generated over $U(\mathfrak{g})$, decompose into direct sums of weight spaces and are locally $U(\mathfrak{p})$ -finite. Denote by P_0 the set of \mathfrak{g}_0 -integral weights, and by P_0^+ the set of \mathfrak{g}_0 -integral dominant weights. Then

$$P_0 = \left\{ \lambda \in \mathfrak{h}^* \mid \frac{2(\lambda, \alpha)}{(\alpha, \alpha)} \in \mathbb{Z}, \forall \alpha \in \Delta_0^+ \right\},$$

$$P_0^+ = \left\{ \lambda \in \mathfrak{h}^* \mid \frac{2(\lambda, \alpha)}{(\alpha, \alpha)} \in \mathbb{Z}_+, \forall \alpha \in \Delta_0^+ \right\},$$

where Δ_0^+ is the set of the positive roots of \mathfrak{g}_0 . Given any $\lambda \in P_0^+$, the simple \mathfrak{p} -module $L^0(\lambda)$ with highest weight λ is necessarily finite dimensional. We define the parabolic Verma module $V(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} L^0(\lambda)$ over $U(\mathfrak{g})$, which has a unique simple quotient $L(\lambda)$. Then $V(\lambda)$ (and hence $L(\lambda)$) belongs to \mathcal{O}^p if and only if $\lambda \in P_0^+$. Furthermore, $\{L(\lambda), \mathfrak{P}(L(\lambda)) \mid \lambda \in P_0^+\}$ is the set of non-isomorphic simple objects in \mathcal{O}^p , where \mathfrak{P} is the parity reversal functor. Set

$$R_{\bar{0}} = \prod_{\alpha \in \Delta_0^+} (e^{\alpha/2} - e^{-\alpha/2}), \quad R_1 = \prod_{\beta \in \Delta_1^+} (e^{\beta/2} + e^{-\beta/2}). \quad (3.1)$$

One can immediately show that the character of the parabolic Verma module $V(\lambda)$ for any $\lambda \in P_0^+$ is given by

$$\text{ch } V(\lambda) = \frac{R_1}{R_{\bar{0}}} \sum_{w \in W_0} \text{sign}(w) e^{w(\lambda + \rho)}. \quad (3.2)$$

Let $P = \left\{ \lambda \in \mathfrak{h}^* \mid \frac{2(\lambda, \alpha)}{(\alpha, \alpha)} \in \mathbb{Z}, \forall \alpha \in \Delta_0^+ \right\}$, and call the elements \mathfrak{g} -integral. The elements of $P \cap P_0^+$ will be said to be \mathfrak{g} -integral and \mathfrak{g}_0 -dominant. Let P^+ be the subset of $P \cap P_0^+$ consisting of elements λ which satisfy the following conditions. Write $\lambda \in P^+$ in coordinates $\lambda = (\lambda_0 \mid \lambda_1, \dots, \lambda_{I_1})$, then

$$\begin{aligned} & \frac{2(\lambda, \alpha)}{(\alpha, \alpha)} \in \mathbb{Z}_+, \quad \forall \alpha \in \Delta_0^+, \quad \text{and} \\ D(2, 1; a) : & \lambda_0 \geq 2 \text{ or } \lambda_0 = 1, \lambda_1 + 1 = \pm a(\lambda_2 + 1) \text{ or } \lambda = 0, \\ F_4 : & \lambda_0 \geq 2 \text{ or } \lambda_0 = \frac{3}{2}, 2(\lambda_1 - \lambda_2 - \lambda_3) + 1 = 0 \text{ or} \\ & \lambda_0 = 1, \lambda_1 = \lambda_2, \lambda_3 = 0 \text{ or } \lambda = 0, \\ G_3 : & \lambda_0 \geq 3 \text{ or } \lambda_0 = 2, \lambda_1 = \lambda_2 \text{ or } \lambda = 0. \end{aligned} \quad (3.3)$$

Call elements of P^+ *integral dominant* with respect to \mathfrak{g} . It is known [21, 22] that

$$\dim L(\lambda) < \infty \quad \text{if and only if } \lambda \in P^+. \quad (3.4)$$

In this case, we let $\bar{\lambda}_0 := \frac{2(\varphi, \lambda)}{(\varphi, \varphi)}$, which is λ_0 if $\mathfrak{g} = D(2, 1; a)$ or G_3 , and $2\lambda_0$ if $\mathfrak{g} = F_4$. Then $U(\mathfrak{g})f_{\varphi}^{\bar{\lambda}_0+1}v_{\lambda}$ is a submodule of $V(\lambda)$; the quotient module

$$K(\lambda) = V(\lambda) / U(\mathfrak{g})f_{\varphi}^{\bar{\lambda}_0+1}v_{\lambda} \quad (3.5)$$

is usually referred to as the *Kac module* [21, 22], which is the maximal finite dimensional quotient of $V(\lambda)$.

An element $\lambda \in P_0$ is *atypical* if there exists some isotropic odd root $\gamma \in \Delta_1^+$ such that $(\lambda^\rho, \gamma) = 0$. In this case γ is called an *atypical root* of λ . If no atypical root exists, λ is called *typical*. Given any atypical weight $\lambda \in P_0$, denote by $\Delta_{\text{aty}}(\lambda) \subset \Delta_1^+$ the set of its atypical roots.

The following result is due to Kac [21, 22].

Proposition 3.1. *Assume that $\lambda \in P^+$ is a typical weight, then $L(\lambda) = K(\lambda)$, and*

$$\text{ch } L(\lambda) = \text{ch } K(\lambda) = \frac{R_1}{R_0} \sum_{w \in W} \text{sign}(w) e^{w(\lambda + \rho)}. \quad (3.6)$$

Recall that a module is said to be rigid if it has a unique Loewy filtration. We have the following result.

Theorem 3.2. *The parabolic Verma modules in $\mathcal{O}^{\mathfrak{p}}$ are rigid.*

The theorem will be proven in Section 6.1.

For any \mathfrak{g} -module V in $\mathcal{O}^{\mathfrak{p}}$, we use V^\vee to denote the \mathfrak{g} -module which is the direct sum of the duals of weight spaces of V with the τ -twisted $U(\mathfrak{g})$ -action defined for all $x \in U(\mathfrak{g})$ and $f \in V^\vee$ by

$$(xf)(v) = f(\tau(x)v), \quad \forall v \in V.$$

Then V^\vee is in $\mathcal{O}^{\mathfrak{p}}$; in particular, $V^\vee = V$ if V is a finite dimensional simple module. This notion of dual modules will be used later.

3.2. Generalised Jantzen filtration. We recall from [32] some elements of the generalised Jantzen filtration for parabolic Verma modules over Lie superalgebras.

Let $T := \mathbb{C}[[t]]$ be the ring of formal power series in the indeterminate t , and consider the category $\mathfrak{g}\text{-Mod-}T$ of \mathbb{Z}_2 -graded $U(\mathfrak{g})$ - T bimodules such that the left action of $\mathbb{C} \subset U(\mathfrak{g})$ and right action of $\mathbb{C} \subset T$ agree. We take the \mathbb{C} -algebra homomorphism $\phi : U(\mathfrak{h}) \rightarrow T$ such that

$$\phi(h) = t\delta(h) \quad \text{for } h \in \mathfrak{h}. \quad (3.7)$$

This defines a \mathfrak{p} -action on T (recall that $\mathfrak{p} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$) by

$$hf = f\phi(h) \quad \text{and} \quad Xf = 0 \quad \text{for } f \in T, h \in \mathfrak{h}, X \in \mathfrak{g}^\alpha \subset \mathfrak{p}.$$

Given any object M in the category $\mathfrak{g}\text{-Mod-}T$, we define the deformed weight space of weight $\mu \in \mathfrak{h}^*$ by

$$M_\mu = \{m \in M \mid hm = \mu(h)m + m\phi(h), \forall h \in \mathfrak{h}\}. \quad (3.8)$$

The deformed parabolic category $\mathcal{O}^{\mathfrak{p}}(T)$ is the full subcategory of $\mathfrak{g}\text{-Mod-}T$ such that each object

- is finitely generated over $U(\mathfrak{g}) \otimes_{\mathbb{C}} T$;
- decomposes into the direct sum of deformed weight spaces; and
- is locally $U(\mathfrak{p}')$ -finite, where $\mathfrak{p}' = [\mathfrak{p}, \mathfrak{p}]$.

The parabolic Verma modules are distinguished objects of $\mathcal{O}^{\mathfrak{p}}(T)$. For $\lambda \in P_0^+$, let $L^0(\lambda)$ be the finite dimensional irreducible \mathfrak{p} -module with highest weight λ . Introduce the \mathfrak{p} -module $L_T^0(\lambda) = L^0(\lambda) \otimes_{\mathbb{C}} T$ with \mathfrak{p} acting diagonally. This is also

a \mathfrak{p} - T -bimodule with T acting on the right by multiplication on the factor T . The parabolic Verma module with highest weight λ is

$$V_T(\lambda) := U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} L_T^0(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} (L^0(\lambda) \otimes_{\mathbb{C}} T). \quad (3.9)$$

Let τ be the anti-involution on the universal enveloping algebra $U(\mathfrak{g})$ defined in (2.7). There exists a unique T -bilinear form

$$\langle \cdot, \cdot \rangle : V_T(\lambda) \times V_T(\lambda) \longrightarrow T \quad (3.10)$$

which is contravariant

$$\langle xm, m' \rangle = \langle m, \tau(x)m' \rangle, \quad \forall m, m' \in V_T(\lambda), x \in U(\mathfrak{g}), \quad (3.11)$$

non-degenerate in the sense that

$$\langle m, V_T(\lambda) \rangle = \{0\} \quad \text{only when } m = 0,$$

and normalised so that the highest weight vector v_λ of $L^0(\lambda)$ satisfies $\langle v_\lambda, v_\lambda \rangle = 1$.

Let $V_T^i(\lambda) = \{m \in V_T(\lambda) \mid \langle m, V_T(\lambda) \rangle \subset t^i \mathbb{C}[[t]]\}$ for each $i \in \mathbb{Z}_+$. These are \mathfrak{g} - T submodules of $V_T(\lambda)$, which give rise to the following descending filtration:

$$V_T(\lambda) = V_T^0(\lambda) \supset V_T^1(\lambda) \supset V_T^2(\lambda) \supset \cdots. \quad (3.12)$$

Regard \mathbb{C} as a T -module with $f(t) \in \mathbb{C}[[t]]$ acting by multiplication by $f(0)$. Let $\mathcal{R} : \mathcal{O}^p(T) \longrightarrow \mathcal{O}^p$ be the specialisation functor which sends an object M in $\mathcal{O}^p(T)$ to $M \otimes_T \mathbb{C}$ in \mathcal{O}^p , and a morphism $\psi : M \rightarrow N$ to

$$\mathcal{R}(\psi) : M \otimes_T \mathbb{C} \rightarrow N \otimes_T \mathbb{C}, \quad \mathcal{R}(\psi)(m \otimes_T c) = \psi(m) \otimes_T c.$$

Denote $V(\lambda) = V_T(\lambda) \otimes_T \mathbb{C}$ and $V^i(\lambda) = V_T^i(\lambda) \otimes_T \mathbb{C}$. Applying the specialisation functor \mathcal{R} to (3.12) we obtain the following descending filtration

$$V(\lambda) = V^0(\lambda) \supset V^1(\lambda) \supset V^2(\lambda) \supset \cdots \quad (3.13)$$

for the parabolic Verma module $V(\lambda)$, which will be referred to as the *Jantzen filtration* for $V(\lambda)$. The consecutive quotients of the Jantzen filtration are

$$V_i(\lambda) = V^i(\lambda) / V^{i+1}(\lambda), \quad i = 0, 1, 2, \dots \quad (3.14)$$

One of the main results on the Jantzen filtration is the following.

Theorem 3.3. *For any $\lambda \in P_0^+$, the Jantzen filtration of the parabolic Verma module $V(\lambda)$ is the unique Loewy filtration.*

We will prove the theorem in Section 6.2.

Let $H_i(\mathfrak{u}^-, L(\lambda))$ be the i -th Lie superalgebra homology group of $\mathfrak{u}^- := \mathfrak{g}_{-1} \oplus \mathfrak{g}_{-2}$ with coefficients in the restriction of $L(\lambda)$. Then $H_i(\mathfrak{u}^-, L(\lambda))$ admits a semi-simple \mathfrak{g}_0 -action. For any $\lambda, \mu \in P_0^+$, define the following *generalised Kazhdan-Lusztig polynomials* [26]:

$$p_{\lambda\mu}(q) = \sum_{i=0}^{\infty} (-q)^i [H_i(\mathfrak{u}^-, L(\lambda)) : L^0(\mu)], \quad (3.15)$$

where $[H_i(\mathfrak{u}^-, L(\lambda)) : L^0(\mu)]$ is the multiplicity of $L^0(\mu)$ in $H_i(\mathfrak{u}^-, L(\lambda))$.

Remark 3.4. The generalised Kazhdan-Lusztig polynomials may also be defined in terms of the \mathfrak{u} -cohomology groups $H^i(\mathfrak{u}, L(\lambda))$ instead [32], as

$$H^i(\mathfrak{u}, L(\lambda)) = H_i(\mathfrak{u}^-, L(\lambda)^\vee) = H_i(\mathfrak{u}^-, L(\lambda)) \quad \text{for all } \lambda \in P_0^+,$$

where $L(\lambda)^\vee$ is the τ -twisted dual of $L(\lambda)$ defined at the end of Section 3.1.

Choose a linear order on P_0^+ compatible with the usual partial order defined by the positive roots. Then the matrix $P(q) = \left(p_{\lambda\mu}(q)\right)_{\lambda, \mu \in P_0^+}$ is upper unitriangular.

Let $A(q) = \left(a_{\lambda\mu}(q)\right)_{\lambda, \mu \in P_0^+}$ be the inverse matrix of $P(q)$, and refer to $a_{\lambda\mu}(q)$ as the *inverse Kazhdan-Lusztig polynomials* of \mathfrak{g} .

For any $\lambda, \mu \in P_0^+$, we also define

$$J_{\lambda\mu}(q) = \sum_{i=0}^{\infty} q^i [V_i(\lambda) : L(\mu)], \quad (3.16)$$

where $[V_i(\lambda) : L(\mu)]$ denotes the multiplicity of the irreducible \mathfrak{g} -module $L(\mu)$ in $V_i(\lambda)$. They were referred to as *Jantzen polynomials* in [32].

Another main result on the Jantzen filtration is the following.

Theorem 3.5. *For any $\lambda, \mu \in P_0^+$, the Jantzen polynomials $J_{\lambda\mu}(q)$ coincide with the inverse Kazhdan-Lusztig polynomials $a_{\lambda\mu}(q)$.*

We will prove the theorem in Section 6.4.

The remainder of the paper is devoted to the proofs of Theorems 3.2, 3.3 and 3.5.

4. CLASSIFICATION OF ATYPICAL WEIGHTS

We introduce some combinatorics for integral atypical weights in this section. In particular, we define two sets P_{aty}^1 and P_{aty}^2 , in each of which weights are related by “up-moves” and “down-moves” defined by Definition 4.3. These two sets partition the set of \mathfrak{g} -integral and \mathfrak{g}_0 -dominant atypical weights. The significance of the combinatorics will become clear in Section 5.3, where it enables us to describe the structure of parabolic Verma modules quite uniformly.

4.1. The up and down moves. We start by observing several simple facts. An element $\lambda = (\lambda_0 \mid \lambda_1, \dots, \lambda_{l_1}) \in \mathfrak{h}^*$ is \mathfrak{g}_0 -integral (resp., \mathfrak{g}_0 -integral dominant) if and only if the following conditions are satisfied:

$$\begin{aligned} D(2, 1; a) : \quad & \lambda_1, \lambda_2 \in \mathbb{Z} \text{ (resp., } \mathbb{Z}_+), \\ F_4 : \quad & 2\lambda_i, \lambda_i - \lambda_j \in \mathbb{Z} \text{ (resp., } \mathbb{Z}_+) \text{ for } 1 \leq i < j, \\ G_3 : \quad & \lambda_1 - \lambda_2, \frac{1}{3}(2\lambda_2 - \lambda_3 - \lambda_1) \in \mathbb{Z} \text{ (resp., } \mathbb{Z}_+). \end{aligned} \quad (4.1)$$

Also, $\lambda \in P_0^+$ is \mathfrak{g} -integral if and only if $\lambda_0 \in \mathbb{Z}$ in the cases of $D(2, 1; a)$ and G_3 , and $2\lambda_0 \in \mathbb{Z}_+$ in the case of F_4 .

Proposition 4.1. (1) *Assume $\mathfrak{g} = D(2, 1; a)$ or F_4 . Let $\lambda \in P_0^+$ be an atypical weight with two atypical roots. Then $\lambda_0^p = 0$; in particular, λ is \mathfrak{g} -integral.*
 (2) *If $\mathfrak{g} = G_3$, then each atypical weight $\lambda \in P_0^+$ has only one atypical root.*

Proof. Assume $\gamma_{\pm} \in \Delta_1^+$ are atypical roots of λ with $\gamma_+ \neq \gamma_-$. First suppose $\mathfrak{g} = D(2, 1; a)$. Then $\lambda_1^\rho, \lambda_2^\rho \geq 1$. Assume $\gamma_{\pm} = \delta + x_{\pm}\varepsilon_1 + y_{\pm}\varepsilon_2$ for some $x_{\pm}, y_{\pm} \in \{\pm 1\}$. Then from $(\lambda^\rho, \gamma_+ - \gamma_-) = 0$, we obtain $x'\lambda_1^\rho + ay'\lambda_2^\rho = 0$, where $x' = x_+ - x_-$, $y' = y_+ - y_- \in \{0, \pm 2\}$ and $(x', y') \neq (0, 0)$. This is impossible if $a \notin \mathbb{Q}$ (the field of rational numbers). If $a \in \mathbb{Q}$, we obtain $x' = \text{sign}(a)y' = \pm 2$ (where $\text{sign}(a)$ is the sign of a), and thus $\gamma_{\pm} = \delta \pm (\varepsilon_1 - \text{sign}(a)\varepsilon_2)$, $\lambda_1^\rho = \text{abs}(a)\lambda_2^\rho$ (where $\text{abs}(a)$ is the absolute value of a) and $\lambda_0^\rho = 0$.

Now assume $\mathfrak{g} = F_4$. Let $\lambda \in P_0^+$ be an atypical weight. Then $\lambda_1^\rho > \lambda_2^\rho > \lambda_3^\rho > 0$. As above, from $(\lambda^\rho, \gamma_+ - \gamma_-) = 0$, we can obtain $x_1\lambda_1^\rho + x_2\lambda_2^\rho + x_3\lambda_3^\rho = 0$ for some $x_i \in \{0, \pm 1\}$. From this, we deduce $x_2 = x_3 = -x_1 = \pm 1$, and $\lambda^\rho = (0 \mid \lambda_1^\rho, \lambda_2^\rho, \lambda_3^\rho)$ with $\gamma_{\pm} = \frac{1}{2}(\delta \pm (\varepsilon_1 - \varepsilon_2 - \varepsilon_3))$. This proves (1).

Assume $\mathfrak{g} = G_3$. We can suppose $\lambda_3 = 0$ and take $\rho = (-\frac{5}{2} \mid 5, 4, 0)$ by (2.12). Then we have $\lambda_1, \lambda_2, \lambda_1 - \lambda_2, \frac{1}{3}(2\lambda_2 - \lambda_1) \in \mathbb{Z}_+$ by (4.1). Thus $\lambda_1^\rho = \lambda_1 + 5 > \lambda_2^\rho = \lambda_2 + 4 > \lambda_3^\rho = 0$. Assume $\gamma_{\pm} = \delta + \varepsilon_{i_{\pm}} - \varepsilon_{j_{\pm}}$ ($1 \leq i_{\pm} \neq j_{\pm} \leq 3$). If $i_- = i_+$ or $j_- = j_+$ or $(i_-, j_-) = (j_+, i_+)$, then from $(\lambda^\rho, \gamma_+ - \gamma_-) = 0$, we would obtain respectively $\lambda_{j_+}^\rho = \lambda_{j_-}^\rho$ ($j_+ \neq j_-$) or $\lambda_{i_+}^\rho = \lambda_{i_-}^\rho$ ($i_+ \neq i_-$) or $\lambda_{i_+}^\rho = \lambda_{j_+}^\rho$ ($i_+ \neq j_+$), a contradiction. Thus $(i_-, j_-) = (j_+, k)$ or (k, i_+) (where i_+, j_+, k are pairwise distinct), but we would then obtain $\lambda_1^\rho = 2\lambda_2^\rho$, which would imply that λ is singular. Thus the atypical root of λ is unique, and (2) is proven. \square

Definition 4.2. An element $\lambda \in P_0$ is called *regular* if there exists $w \in W_0$ such that $w \cdot \lambda \in P_0^+$, and in this case, denote $\lambda^+ = w \cdot \lambda$. If λ is not regular, it is called *singular*.

Definition 4.3. Assume that $\lambda \in P_0$ has an atypical root $\gamma \in \Delta_1^+$. Let k (resp. k') be the smallest positive integer rendering $\lambda + k\gamma$ (resp., $\lambda - k'\gamma$) regular, and define

$$\lambda_{\gamma} = (\lambda + k\gamma)^+, \quad \check{\lambda}_{\gamma} = (\lambda - k'\gamma)^+ \quad \text{in } P_0^+.$$

Call the procedure of obtaining λ_{γ} (resp. $\check{\lambda}_{\gamma}$) from λ an *up* (resp. *down*) *move along* γ . If γ is the only atypical root of λ , we simply write $\check{\lambda} = \check{\lambda}_{\gamma}$, $\lambda = \lambda_{\gamma}$.

Remark 4.4. For any $\nu \in \mathfrak{h}^*$, we denote by χ_{ν} the central character determined by ν . If ν is typical, then $\chi_{\mu} = \chi_{\nu}$ if and only if $\mu = w \cdot \nu$ for some $w \in W$. If ν is atypical, then $\chi_{\mu} = \chi_{\nu}$ if and only if there exist atypical elements $\mu_i \in \mathfrak{h}^*$, $\gamma_i \in \Delta_{\text{aty}}(\mu_i)$, $t_i \in \mathbb{C}$, and $w_i \in W$ with $i = 0, 1, \dots, k$ for some $k < \infty$ such that

$$\mu_0 = \nu, \quad \mu_{i+1} = w_i \cdot (\mu_i + t_i \gamma_i) \quad \text{for all } i < k, \quad \mu_{k+1} = \mu.$$

Remark 4.5. It follows from Remark 4.4 that both λ_{γ} and $\check{\lambda}_{\gamma}$ correspond to the same central character as λ .

Remark 4.6. Repeated applications of up (resp. down) moves to λ produce the weights $(\lambda + k\gamma)^+$ (resp. $(\lambda - k'\gamma)^+$) for all $k > 0$ such that $\lambda + k\gamma$ (resp. $\lambda - k'\gamma$) are regular.

4.2. Description of P_{aty}^1 and P_{aty}^2 . Now we define P_{aty}^1 and P_{aty}^2 for the exceptional Lie superalgebras case by case.

4.2.1. The case $D(2, 1; a)$. Take $\mu = -\rho = (1 \mid -1, -1)$ (i.e., $\mu^\rho = 0$), which is a singular atypical weight with an atypical root $\gamma = \delta - \varepsilon_1 - \varepsilon_2$ (in fact one can take

γ to be any root in Δ_1^+). We set $\lambda^1 = \mu^\gamma = (2 \mid 0, 0)$, $\lambda^{-1} = \mu^\gamma = 0$. In general, for any $i \in \mathbb{Z}^* := \mathbb{Z} \setminus \{0\}$, we let

$$\lambda^i = \begin{cases} (i+1 \mid i-1, i-1), & \text{if } i > 0, \\ (i+1 \mid -i-1, -i-1), & \text{if } i < 0, \end{cases} \quad (4.2)$$

and denote $P_{\text{aty}}^1 = \{\lambda^i \mid i \in \mathbb{Z}^*\}$. Then one has

$$(\lambda^i)^\gamma = \lambda^{i+1} \ (i \neq -1), \quad (\lambda^{-1})^\gamma = \lambda^1, \quad (\lambda^i)^{\sigma_0} = \lambda^{-i}, \quad (\lambda^\gamma)^\gamma = \lambda. \quad (4.3)$$

[There is a slight difference between the notation here and that in [33]. Here λ^0 is undefined and the weight λ^i for $i \leq -1$ corresponds to $\lambda^{(i+1)}$ in [33].]

If $a \notin \mathbb{Q}$, we set $P_{\text{aty}}^2 = \emptyset$. Now assume $a = \frac{p}{q}$ for some coprime integers $p \neq 0$ and $q > 0$ (and $p \neq -q$). For any fixed $x \in \mathbb{Z}_+ \setminus \{0\}$, we take $\lambda_\pm^0 = \lambda^0$ to be the weight such that

$$(\lambda^0)^\rho = (0 \mid \text{abs}(p)x, qx), \quad \text{i.e., } \lambda^0 = (1 \mid \text{abs}(p)x - 1, qx - 1), \quad (4.4)$$

which is a \mathfrak{g} -integral dominant atypical weight with two atypical roots $\gamma_\pm = \delta \pm (\varepsilon_1 - \text{sign}(p)\varepsilon_2)$. We use Definition 4.3 to define

$$\lambda_\pm^1 = (\lambda^0)^\gamma_{\gamma_\pm}, \quad \lambda_\pm^{-1} = (\lambda^0)^\gamma_{\gamma_\mp}. \quad (4.5)$$

Then $\lambda_+^1 = (\lambda_+^{-1})^{\sigma_0}$, $\lambda_-^1 = (\lambda_-^{-1})^{\sigma_0}$, and

$$\lambda_+^1 = \begin{cases} (2 \mid \text{abs}(p)x, qx - \text{sign}(p) - 1) & \text{if } x > 1 \text{ or } q > 1 \text{ or } p < 0, \\ (3 \mid p+1, 0) & \text{if } x = q = 1, p > 0, \end{cases} \quad (4.6)$$

$$\lambda_-^1 = \begin{cases} (2 \mid \text{abs}(p)x - 2, qx + \text{sign}(p) - 1) & \text{if } x > 1 \text{ or } p > 1 \text{ or } p < -1, q > 1, \\ (3 \mid 1, q + 2\text{sign}(p)) & \text{if } x = p = 1 \text{ or } x = -p = 1, q > 2, \\ (4 \mid 1, 0) & \text{if } x = -p = 1, q = 2. \end{cases}$$

Now for $i \geq 2$, we define

$$\lambda_\pm^i = (\lambda_\pm^{i-1})^\gamma, \quad \lambda_\pm^{-i} = (\lambda_\pm^{1-i})^\gamma. \quad (4.7)$$

Then λ_\pm^i 's are \mathfrak{g} -integral and \mathfrak{g}_0 -dominant atypical weights and $(\lambda_\pm^i)^{\sigma_0} = \lambda_\pm^{-i}$. We let P_{aty}^2 be the set consisting of the weights $\lambda^0 = \lambda_\pm^0$ and λ_\pm^i ($i \in \mathbb{Z}^*$) for all $x \in \mathbb{Z}_+ \setminus \{0\}$.

4.2.2. *The case F_4 .* Let $x \in \mathbb{Z}_+ \setminus \{0\}$ be fixed, and set (cf. (4.4))

$$\mu^\rho = (0 \mid x, x, 0), \quad \text{i.e., } \mu = \left(\frac{3}{2} \mid x - \frac{5}{2}, x - \frac{3}{2}, \frac{1}{2}\right). \quad (4.8)$$

Note that μ is a singular atypical weight with an atypical root $\gamma = \frac{1}{2}(\delta + \varepsilon_1 - \varepsilon_2 - \varepsilon_3)$ (in fact one can take γ to be any $\frac{1}{2}(\delta \pm (\varepsilon_1 - \varepsilon_2) \pm \varepsilon_3) \in \Delta_1^+$ and obtain the same λ^i). Set $\lambda^1 = \mu^\gamma$, $\lambda^{-1} = \mu^\gamma$, which are given by

$$\lambda^1 = \begin{cases} (2 \mid x-2, x-2, 0) & \text{if } x \geq 2, \\ (3 \mid 0, 0, 0) & \text{if } x = 1, \end{cases} \quad (4.9)$$

$$\lambda^{-1} = \begin{cases} (1 \mid x-2, x-2, 0) & \text{if } x \geq 2, \\ 0 & \text{if } x = 1. \end{cases}$$

For all $i \geq 2$, let $\lambda^i = (\lambda^{i-1})^\gamma$, $\lambda^{-i} = (\lambda^{1-i})^\gamma$ (note that λ^0 is not defined). Then (4.3) holds. Let P_{aty}^1 be the set of all λ^i with $i \in \mathbb{Z}^*$ and $x \in \mathbb{Z}_+ \setminus \{0\}$.

Let $1 \leq a_3 < a_2 \in \mathbb{Z}_+$ be fixed integers and set $a_1 = a_2 + a_3$ (thus $a_2 \geq a_3 + 1 \geq 2$ and $a_1 \geq 3$). We take $\lambda_\pm^0 = \lambda^0$ to be the weight such that (cf. (4.4) and (4.8))

$$(\lambda^0)^\rho = (0 \mid a_1, a_2, a_3), \quad \text{i.e., } \lambda^0 = \left(\frac{3}{2} \mid a_1 - \frac{5}{2}, a_2 - \frac{3}{2}, a_3 - \frac{1}{2}\right), \quad (4.10)$$

which is an atypical \mathfrak{g} -integral dominant weight with two atypical roots $\gamma_\pm := \frac{1}{2}(\delta \pm (\varepsilon_1 - \varepsilon_2 - \varepsilon_3))$. We define $\lambda_\pm^{\pm 1}$ as in (4.5). Then

$$\begin{aligned} \lambda_+^1 &= (2 \mid a_1 - 2, a_2 - 2, a_3 - 1), \\ \lambda_+^{-1} &= (1 \mid a_1 - 2, a_2 - 2, a_3 - 1), \\ \lambda_-^1 &= \begin{cases} (2 \mid a_1 - 3, a_2 - 1, a_3) & \text{if } a_3 \geq 2, \\ (\frac{5}{2} \mid a_2 - \frac{3}{2}, a_2 - \frac{3}{2}, \frac{1}{2}) & \text{if } a_3 = 1, \end{cases} \\ \lambda_-^{-1} &= \begin{cases} (1 \mid a_1 - 3, a_2 - 1, a_3) & \text{if } a_3 \geq 2, \\ (\frac{1}{2} \mid a_2 - \frac{3}{2}, a_2 - \frac{3}{2}, \frac{1}{2}) & \text{if } a_3 = 1. \end{cases} \end{aligned} \quad (4.11)$$

Now define $\lambda_\pm^{\pm i}$ for $i \geq 2$ by (4.7), and let $P_{\text{aty}}^2 = \bigcup (\{\lambda^0 = \lambda_\pm^0\} \cup \{\lambda_\pm^i \mid i \in \mathbb{Z}^*\})$, where the union is over all a_i satisfying the given condition.

4.2.3. The case G_3 . As in the proof of Proposition 4.1, we always assume $\lambda_3 = 0$ for any weight λ . Fix $x \in \mathbb{Z}_+$, similar to (4.8), we denote

$$\mu^\rho = \left(\frac{1}{2} \mid 3x + 2, 3x + 1, 0\right), \quad \text{i.e., } \mu = (3 \mid 3x - 3, 3x - 3, 0), \quad (4.12)$$

which is an atypical weight with the unique atypical root $\gamma = \delta + \varepsilon_1 - \varepsilon_2$. If $x \geq 1$, then μ is a \mathfrak{g} -integral dominant weight, and we set $\lambda^1 = \mu$. If $x = 0$, then $\mu, \mu + \gamma$ are singular (cf. Definition 4.2), and we set $\lambda^1 = \hat{\mu} = (\mu + 2\gamma)^+ = (5 \mid 0, 0, 0)$. Define $\lambda^{-1} := (\lambda^1)^{\sigma_0} = (\lambda^1)^\sim = (2 \mid 3x - 3, 3x - 3, 0)$ if $x \geq 1$ or $\lambda^{-1} = 0$ otherwise.

For $i \geq 2$, we define $\lambda^i = (\lambda^{i-1})^\sim$ and $\lambda^{-i} = (\lambda^{1-i})^\sim$ (there is no λ^0). Then (4.3) holds. The following fact will be needed later:

$$\lambda_0^i > \lambda_0^1 \geq 3 \quad \text{for all } i \geq 2. \quad (4.13)$$

Denote by P_{aty}^1 the set of \mathfrak{g} -integral and \mathfrak{g}_0 -dominant atypical weights obtained in this way for all $x \in \mathbb{Z}_+$, and set $P_{\text{aty}}^2 = \emptyset$.

4.3. A classification of integral atypical weights. Let \mathfrak{g} be an exceptional Lie superalgebra. We have the following classification of \mathfrak{g} -integral atypical weights.

Proposition 4.7. *Let $P_{\text{aty}} := P_{\text{aty}}^1 \cup P_{\text{aty}}^2$. Then every \mathfrak{g} -integral and \mathfrak{g}_0 -dominant atypical weight belongs to P_{aty} . Furthermore, a weight $\lambda^i \in P_{\text{aty}}^1$ is \mathfrak{g} -integral dominant if and only if $i = -1$ or $i \geq 1$, and a weight $\lambda_\pm^i \in P_{\text{aty}}^2$ is \mathfrak{g} -integral dominant if and only if $i \geq 0$.*

Proof. Suppose λ is a \mathfrak{g} -integral and \mathfrak{g}_0 -dominant atypical weight with an atypical root $\gamma' \in \Delta_1^+$. Then $\lambda_0 \in \mathbb{Z}$ if $\mathfrak{g} = D(2, 1; a)$ or G_3 , and $\lambda_0 \in \frac{1}{2}\mathbb{Z}$ otherwise. Note that there exists a unique $k \in \mathbb{Z}$ such that the 0-th coordinate of $\nu^\rho := \lambda^\rho + k\gamma$ is $\nu_0^\rho = 0$ (if $\mathfrak{g} = D(2, 1; a)$ or F_4) or $\nu_0^\rho = \frac{1}{2}$ (if $\mathfrak{g} = G_3$). Then either $\mu := \nu = \nu^\rho - \rho$ is singular with atypical root γ' , or there exists a unique $\sigma \in W_0$ such that $\mu := \sigma(\nu^\rho) - \rho$ is \mathfrak{g}_0 -integral dominant with atypical root $\gamma = \sigma(\gamma')$. Thus μ is one of the weights used

to generate elements in P_{aty}^1 or P_{aty}^2 . It then follows from Remark 4.6 that $\lambda \in P_{\text{aty}}$. The second statement can be easily proven by inspecting (3.3). \square

Denote by $P_{\text{aty}}^+ = P_{\text{aty}} \cap P^+$ the set of \mathfrak{g} -integral dominant atypical weights. Let

$$P_{\text{aty}}^t = \text{the set of all } \lambda^i, \lambda_{\pm}^i \in P_{\text{aty}} \text{ with } i < 0. \quad (4.14)$$

Every atypical weight λ in P_{aty}^t satisfies $\lambda_0^\rho < 0$, and in this case every $\gamma \in \Delta_{\text{aty}}(\lambda)$ is called a *tail atypical root* of λ (note that $\gamma \in \Delta_1^\pm$ if $\lambda \in P_{\text{aty}}^1$ or $\lambda = \lambda_+^i \in P_{\text{aty}}^2$). For $\lambda \in P_{\text{aty}} \setminus P_{\text{aty}}^t$, an atypical root $\gamma \in \Delta_{\text{aty}}(\lambda)$ is called a *non-tail atypical root* of λ .

5. STRUCTURE OF PARABOLIC VERMA MODULES

Let \mathfrak{g} be an exceptional Lie superalgebra with the distinguished maximal parabolic subalgebra \mathfrak{p} as defined in Section 3.1.

5.1. Primitive weight graphs. We briefly recall from [32] (see also [29]) the notion of primitive weight graphs, which is needed in later sections.

Let V be an object in $\mathcal{O}^{\mathfrak{p}}$. A nonzero \mathfrak{g}_0 -highest weight vector $v \in V$ is called a *primitive vector* if there exists a \mathfrak{g} -submodule M of V such that $v \notin M$ but $uv \in M$. If we can take $M = 0$, then v is a \mathfrak{g} -highest weight vector. The weight of a primitive vector is called a *primitive weight*. For a primitive weight μ of a \mathfrak{g} -module V , we shall use v_μ to denote a nonzero primitive vector of weight μ which generates an indecomposable submodule. Two primitive vectors are regarded as the same if they generate the same indecomposable submodule.

Denote by $P(V)$ the multi-set of primitive weights of V , where the multiplicity of a primitive weight μ is equal to the dimension of the subspace spanned by all the primitive vectors with weight μ .

For $\mu, \nu \in P(V)$, if $\mu \neq \nu$ and $v_\nu \in U(\mathfrak{g})v_\mu$, we say that ν is *derived from* μ and write $\nu \leftarrow \mu$ or $\mu \rightarrow \nu$. If $\mu \rightarrow \nu$ and there exists no $\lambda \in P(V)$ such that $\mu \rightarrow \lambda \rightarrow \nu$, then we say that ν is *directly derived from* μ and write $\mu \rightarrow \nu$ or $\nu \leftarrow \mu$.

Definition 5.1. [32] We associate $P(V)$ with a directed graph, still denoted by $P(V)$, in the following way: the vertices of the graph are elements of the multi-set $P(V)$ (i.e., a primitive weight of multiplicity m corresponds to m distinct vertices). Two vertices λ and μ are connected by a single directed edge pointing toward μ if and only if μ is derived from λ . We shall call this graph the *primitive weight graph of* V .

The *skeleton* of the primitive weight graph is the subgraph containing all the vertices and is such that two vertices λ and μ are connected by a single directed edge pointing to μ if and only if μ is directly derived from λ . In this case we say that the two weights are linked.

Note that a primitive weight graph is uniquely determined by its skeleton.

A *full subgraph* S of $P(V)$ is a subset of $P(V)$ which contains all the edges linking vertices of S . If for any $\mu, \nu \in S$, we have that $\mu \rightarrow \eta \rightarrow \nu$ implies $\eta \in S$, we call the full subgraph S *closed*. It is clear that a module is indecomposable if and only if its primitive weight graph is *connected* (in the usual sense), and that a full subgraph of $P(V)$ corresponds to a subquotient of V if and only if it is closed.

Notation 5.2. For a directed graph Γ , we denote by $M(\Gamma)$ any module with primitive weight graph Γ if such a module exists.

Observe the following facts: If Γ is a closed full subgraph of $P(V)$, then $M(\Gamma)$ always exists, which is a subquotient of V . Also, the primitive weight graph of V^\vee is obtained from that of V by reversing the directions of the edges.

Remark 5.3. We may regard $P(V)$ as a set in such a way that any member λ of multiplicity $m_\lambda > 1$ will be considered as m_λ distinct elements.

5.2. The \mathfrak{g}_0 -highest weights in parabolic Verma modules. We describe the set of \mathfrak{g}_0 -highest weights in the atypical parabolic Verma module $V(\lambda)$. This contains the set $P(V(\lambda))$ of primitive weights of $V(\lambda)$.

Remark 5.4. We will show presently that every parabolic Verma module $V(\lambda)$ in \mathcal{O}^p is multiplicity free, namely, $\mu_\lambda = 1$ for all $\lambda \in P(V)$.

Given $\lambda \in P_0^+$, we define

$$P_\lambda^+ := \{\mu \in P_0^+ \mid \mu \preceq \lambda, \chi_\mu = \chi_\lambda\}. \quad (5.1)$$

If M is a highest weight module for \mathfrak{g} with highest weight λ , let

$$P_0(M) := \{\mu \in P_\lambda^+ \mid \mu \text{ is a } \mathfrak{g}_0\text{-highest weight in } M\}. \quad (5.2)$$

Then $P(V(\lambda)) \subset P_0(V(\lambda)) \subset P_\lambda^+$ as every primitive weight μ of $V(\lambda)$ must correspond to the same central character as λ does itself, i.e., $\chi_\mu = \chi_\lambda$. Let

$$a_{\lambda,\mu} = [V(\lambda) : L(\mu)], \quad b_{\lambda,\mu} = [V(\lambda) : L^0(\mu)], \quad (5.3)$$

where $V(\lambda)$ is regarded as a \mathfrak{g}_0 -module by restriction in the second formula. Clearly $a_{\lambda,\mu} \leq b_{\lambda,\mu}$ for all $\mu \in P(V(\lambda))$.

In the following discussion, when $\lambda \in P_{\text{aty}}^2$, we may assume $\lambda = \lambda_+^i$ for some i as the case $\lambda = \lambda_-^i$ is exactly the same. Note that symbols $\lambda^0, \lambda_+^0, \lambda_-^0$ all denote the same weight $\lambda^0 \in P_{\text{aty}}^2$. The following simple facts will be frequently used (cf. (4.3) and (4.7)): for all i ,

$$\begin{aligned} \lambda^\vee &= \begin{cases} \lambda^{i-1} & \text{if } \lambda = \lambda^i \in P_{\text{aty}}^1 \text{ with } i \neq 0, 1, \\ \lambda^{-1} & \text{if } \lambda = \lambda^1 \in P_{\text{aty}}^1, \\ \lambda_\pm^{i-1} & \text{if } \lambda = \lambda_\pm^i \in P_{\text{aty}}^2 \text{ with } i \neq 0, \end{cases} \\ \lambda^{\sigma_0} &= \begin{cases} \lambda^{-i} & \text{if } \lambda = \lambda^i \in P_{\text{aty}}^1, \\ \lambda_\pm^{-i} & \text{if } \lambda = \lambda_\pm^i \in P_{\text{aty}}^2. \end{cases} \end{aligned} \quad (5.4)$$

Lemma 5.5. *Let $\lambda \in P_{\text{aty}}$. Then*

(1)

$$P_\lambda^+ = \begin{cases} \{\lambda^j \mid 0 \neq j \leq i\} & \text{if } \lambda = \lambda^i \in P_{\text{aty}}^1, \\ \{\lambda_+^j \mid j \leq i\} \cup \{\lambda_-^k \mid k < 0\} & \text{if } \lambda = \lambda_+^i \in P_{\text{aty}}^2 \text{ with } i \geq 0, \\ \{\lambda_+^j \mid j \leq i\} & \text{if } \lambda = \lambda_+^i \in P_{\text{aty}}^2 \text{ with } i < 0, \end{cases}$$

and similarly for $\lambda = \lambda_-^i \in P_{\text{aty}}^2$.

- (2) • if $\lambda = \lambda^0 \in P_{\text{aty}}^2$, then $P_0(V(\lambda)) \subset \{\lambda^0, \lambda_\pm^{-1}\}$ and $b_{\lambda^0, \lambda_\pm^{-1}} \leq 1$;
 • if $\lambda = \lambda^1 \in P_{\text{aty}}^1$, then $P_0(V(\lambda)) \subset \{\lambda^1, \lambda^{-1}, \lambda^{-2}\}$;

- if λ is not as above, let $\Omega_\lambda := \{\lambda, \check{\lambda}, \lambda^{\sigma_0}, (\lambda^{\sigma_0})^\check{\cdot}, (\lambda^{\sigma_0})^\cdot\}$. Then (cf. (5.4))

$$P_0(V(\lambda)) \subset \Omega_\lambda, \quad \text{and } b_{\lambda, \check{\lambda}} \leq 1. \quad (5.5)$$

In particular, if $\lambda \neq \lambda^0$ has a tail atypical root, then $P_0(V(\lambda)) \subset \{\lambda, \check{\lambda}\}$.

Proof. (1) Part (1) can be verified directly by using Remark 4.4 and definitions of λ^i, λ_\pm^i .

(2) It follows from the PBW Theorem that $U(\mathfrak{u}^-) = U(\mathfrak{g}_{-1} \oplus \mathfrak{g}_{-2})$ has a basis

$$B = \left\{ f_\Theta = f_\varphi^{\theta_0} \prod_{\alpha \in \Delta_1^+} f_\alpha^{\theta_\alpha} \mid \Theta \in \mathbb{Z}_+ \times \{0, 1\}^r \right\}, \quad (5.6)$$

where $r = \#\Delta_1^+$, $\Theta = \{\theta_0, \theta_\alpha\}_{\alpha \in \Delta_1^+}$, and the product in f_Θ is ordered so that f_α is placed before f_β if $\alpha > \beta$ for any $\alpha, \beta \in \Delta_1^+$. Define a total order on B by

$$f_\Theta > f_{\Theta'} \iff |\Theta| > |\Theta'| \text{ or } |\Theta| = |\Theta'| \text{ but } \Theta > \Theta',$$

where $|\Theta| = \theta_0 + \sum_{\alpha \in \Delta_1^+} \theta_\alpha$ is the *level* of Θ , and $\mathbb{Z}_+ \times \{0, 1\}^r$ is ordered lexicographically. Recall that a nonzero vector $v \in V(\lambda)$ can be uniquely written as

$$v = b_1 v_1 + \cdots + b_t v_t, \quad b_i \in B, \quad b_1 > b_2 > \cdots, \quad 0 \neq v_i \in L^0(\lambda). \quad (5.7)$$

We call $b_1 v_1$ the *leading term* (cf. [29, §5]). A term $b_i v_i$ is called a *prime term* if $v_i \in \mathbb{C}v_\lambda$. Note that a vector v may have zero or more than one prime terms. One can immediately prove the following facts (cf. [29, Lemmas 5.1 and 5.2] and [33, Lemmas 3.5 and 3.6]).

Fact 1. Let $v = gu$ for some $u \in V(\lambda)$ and $g \in U(\mathfrak{u}^-)$.

(1) If u has no prime term then v has no prime term.

(2) Let $v' = gu', u' \in V(\lambda)$. If u, u' have the same prime terms then v, v' have the same prime terms.

2. Let $v_\mu \in V(\lambda)$ be a \mathfrak{g}_0 -highest weight vector of weight μ . Then

$$\lambda - \mu = \theta_0 \varphi + \sum_{\alpha \in \Delta_1^+} \theta_\alpha \alpha, \quad (5.8)$$

for some $\Theta = \{\theta_0, \theta_\alpha\}_{\alpha \in \Delta_1^+} \in \mathbb{Z}_+ \times \{0, 1\}^r$. Furthermore, the leading term $b_1 v_1$ of v_μ must be a prime term.

3. Suppose $v'_\mu = \sum_{i=1}^{t'} b'_i v'_i$ is another \mathfrak{g}_0 -highest weight vector with weight μ . If all prime terms of v_μ are the same as those of v'_μ , then $v_\mu = v'_\mu$.

For any given $\mu \in P_0(V(\lambda))$, it follows from (5.8) that

$$\begin{aligned} \text{abs}(\lambda_i - \mu_i) &\leq 2, \quad i = 1, \dots, I_1 \quad \text{if } \mathfrak{g} = D(2, 1; a) \text{ or } F_4, \\ \text{abs}(\lambda_i - \mu_i) &\leq 3, \quad i = 1, 2 \quad \text{if } \mathfrak{g} = G_3. \end{aligned} \quad (5.9)$$

For $\mathfrak{g} = G_3$, we always assume that $\lambda_3 = \mu_3 = 0$ (cf. (2.12)), and when an odd positive root like $\alpha = \delta + \varepsilon_1 - \varepsilon_3$ appears in the right-hand side of (5.8), we change it to $\delta + 2\varepsilon_1 + \varepsilon_2$, as both represent the same weight by (2.12). From (5.9), we can verify directly (case by case) that $\mu \in \{\lambda^0, \lambda_\pm^{-1}\}$ if $\lambda = \lambda^0 \in P_{\text{aty}}^2$, and $\mu \in \Omega_\lambda$ otherwise. Furthermore, in each of the following three cases: (i) $\lambda^1 \neq \lambda \in P_{\text{aty}}^1$ and $\mu = \check{\lambda}$, (ii) $\lambda^0 \neq \lambda \in P_{\text{aty}}^2$ and $\mu = \check{\lambda}$, (iii) $\lambda = \lambda^0 \in P_{\text{aty}}^2$ and $\mu = \lambda_\pm^{-1}$ (in all these cases, $\lambda_0^\rho, \mu_0^\rho$ have the same sign or λ_0^ρ is zero), the Θ in (5.8) is unique. By Fact 3, we obtain $b_{\lambda, \mu} \leq 1$. This proves Lemma 5.5(2). \square

5.3. Structure theorem for parabolic Verma modules. Now we prove the structure theorem of parabolic Verma modules. Recall that P_0^+ (resp. P^+) is the set of weights which are integral dominant with respect to \mathfrak{g}_0 (resp. \mathfrak{g}).

Theorem 5.6. *Let \mathfrak{g} be an exceptional Lie superalgebra, and $V(\lambda)$ be the parabolic Verma module with highest weight $\lambda \in P_0^+$.*

- (1) *Assume that λ is typical. Then $V(\lambda)$ is irreducible if $\lambda \notin P^+$, or has the primitive weight graph $\lambda \rightarrow \lambda^{\sigma_0}$ if $\lambda \in P^+$.*
- (2) *Assume that λ is atypical. If $\lambda \notin P^+$, or $\lambda \in P^+$ but has a tail atypical root, then $V(\lambda)$ has the primitive weight graph $\lambda \rightarrow \lambda^\sim$.*
- (3) *Assume that $\lambda \in P^+$ is atypical with a non-tail atypical root (i.e., $\lambda = \lambda^i \in P_{\text{aty}}^1$ with $i \geq 1$ or $\lambda = \lambda_{\pm}^i \in P_{\text{aty}}^2$ with $i \geq 0$), then the skeleton of the primitive weight graph for $V(\lambda)$ is one of the following directed graphs (cf. (5.4) for symbols appearing in the last graph):*

$$\begin{array}{c} \lambda = \lambda^0 \in P_{\text{aty}}^2 \\ \swarrow \quad \searrow \\ \lambda_+^{-1} \quad \lambda_-^{-1} \end{array}, \quad \begin{array}{c} \lambda = \lambda^1 \in P_{\text{aty}}^1 \\ \downarrow \\ \lambda^{-2} \end{array}, \quad \begin{array}{c} \lambda = \lambda^2 \in P_{\text{aty}}^1 \\ \downarrow \quad \searrow \\ \lambda^{-1} \quad \lambda^1 \\ \swarrow \quad \searrow \\ \lambda^{-3} \quad \lambda^{-2} \end{array}, \quad \begin{array}{c} \lambda \\ \swarrow \quad \searrow \\ \lambda^{\sigma_0} \quad \lambda^\sim \\ \swarrow \quad \searrow \\ (\lambda^{\sigma_0})^\sim \end{array}. \quad (5.10)$$

$\lambda = \lambda^i \in P_{\text{aty}}^1$ with $i \geq 3$, or
 $\lambda = \lambda_{\pm}^i \in P_{\text{aty}}^2$ with $i \geq 1$

Proof. (1) If λ is typical and \mathfrak{g} -integral dominant, then using Remark 4.4, one obtains $P_\lambda^+ = \{\lambda, \lambda^{\sigma_0}\}$ from the definition (5.1). Since $V(\lambda)$ has at least two composition factors (see (3.5)), λ^{σ_0} appears in $P(V(\lambda))$. Now we determine its multiplicity. By (3.5) and Proposition 3.1, the maximal submodule of $V(\lambda)$ is generated by $w_1 := f_\varphi^{\lambda_0+1} v_\lambda$. So, a primitive vector with weight λ^{σ_0} has the following form

$$v'_{\lambda^{\sigma_0}} = u w_1 = u f_\varphi^{\bar{\lambda}_0+1} v_\lambda \quad \text{for some } u \in U(\mathfrak{g}). \quad (5.11)$$

Decompose $U(\mathfrak{g})$ into $U(\mathfrak{g}) = U(\mathfrak{g}^-)U(\mathfrak{g}_1)U(\mathfrak{g}_0^{\geq 0})$, where $\mathfrak{g}^- = \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}^{-\alpha}$, $\mathfrak{g}_0^{\geq 0} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta_0^+} \mathfrak{g}^\alpha$, and where we have adopted the convention that for any subspace M of \mathfrak{g} , we use $U(M)$ to denote the subspace of $U(\mathfrak{g})$ spanned by PBW-monomials with respect to a fixed ordered basis of M . By (5.11) and the fact that w_1 is a \mathfrak{g}_0 -highest weight vector, we can choose $u \in U(\mathfrak{g}^-)U(\mathfrak{g}_1)$. Since w_1 has weight $\mu := \lambda - (\bar{\lambda}_0 + 1)\varphi$, and $\lambda^{\sigma_0} - \mu = \sum_{\alpha \in \Delta_1^+} \alpha = 2\rho_1$, which is the maximal weight of $U(\mathfrak{g}_1) = \wedge \mathfrak{g}_1$, we see that u has to be in $U(\mathfrak{g}_1)$ with weight $2\rho_1$, i.e.,

$$u = \prod_{\alpha \in \Delta_1^+} e_\alpha \quad \text{up to a nonzero scalar factor,} \quad (5.12)$$

where the order of the product is as specified in (5.6). [Actually the order does not matter.] This proves $v'_{\lambda^{\sigma_0}}$ is unique, i.e., we have the graph $\lambda \rightarrow \lambda^{\sigma_0}$ in this case.

If λ is typical but not \mathfrak{g} -integral dominant, then either $P_\lambda^+ = \{\lambda\}$, or $P_\lambda^+ = \{\lambda, \lambda^{\sigma_0}\}$ if $\mathfrak{g} = G_3$ and with λ_0 being a half integer (otherwise $\lambda - \lambda^{\sigma_0} \notin \mathbb{Z}_+\Pi$, cf. (2.17)). In the latter case, λ is not \mathfrak{g} -integral, and one can verify that $V(\lambda)$ does not have a \mathfrak{g}_0 -highest weight vector with weight λ^{σ_0} . This proves (1).

(2) Next assume that λ is atypical but not \mathfrak{g} -integral dominant, or λ is a \mathfrak{g} -integral dominant atypical weight with a tail atypical root. Note from the proof of Lemma 5.5 that (5.5) holds even if λ is not \mathfrak{g} -integral. One can verify that either $V(\lambda)$ does not have a \mathfrak{g}_0 -highest weight vector with weight in $\Omega_\lambda \setminus \{\lambda, \lambda\}$, or elements in $\Omega_\lambda \setminus \{\lambda, \lambda\}$ are not $\preceq \lambda$. Thus $P(V(\lambda)) \subset \{\lambda, \lambda\}$. Let $w_2 := \prod_{\alpha \in \Delta_1^+} f_\alpha v_\lambda$ (with product being ordered as in (5.6)), which has weight $\lambda - 2\rho_1$. One can prove that up to a nonzero scalar multiple (cf. [32, Equation (2.9)]),

$$\prod_{\alpha \in \Delta_1^+} e_\alpha w_2 = \prod_{\alpha \in \Delta_1^+} e_\alpha \prod_{\alpha \in \Delta_1^+} f_\alpha v_\lambda = \prod_{\alpha \in \Delta_1^+} (\lambda + \rho, \alpha) v_\lambda = 0, \quad (5.13)$$

where the last equality follows from the atypicality of λ . This implies that there is a \mathfrak{g} -highest weight vector with weight $\prec \lambda$. Indeed, let $u \in U(\mathfrak{g}_1)$ be the element with a maximal weight such that $w_3 := uw_2 \neq 0$, then w_3 is a \mathfrak{g} -highest weight vector. Since $P(V(\lambda)) \subset \{\lambda, \lambda\}$, the weight of w_3 must be λ . Thus $1 \leq a_{\lambda, \lambda} \leq b_{\lambda, \lambda} \leq 1$, and we have the graph $\lambda \rightarrow \lambda$. This proves (2).

(3) Finally assume λ is an atypical \mathfrak{g} -integral dominant weight with a non-tail atypical root. Then $\lambda = \lambda^i \in P_{\text{aty}}^1$ with $i \geq 1$ or $\lambda = \lambda_\pm^i \in P_{\text{aty}}^2$ with $i \geq 0$. For any \mathfrak{g}_0 -integral dominant weights λ, μ with $\mu \preceq \lambda$, we obtain from the character formulae for $\text{ch } V(\lambda)$ and $\text{ch } L^0(\mu)$ that

$$b_{\lambda, \mu} = \sum_{(S, p, w)} \text{sign}(w), \quad (5.14)$$

where the sum is over all triples $(S, p, w) \in \{S \subset \Delta_1^+\} \times \mathbb{Z}_+ \times W_0$ such that $\mu = w \cdot \nu$ with $\nu := \lambda - \sum_{\alpha \in S} \alpha - p\varphi$ being regular. Note that these conditions in particular imply $\#S = |\lambda - \mu| - p$, where $|\lambda - \mu|$ is the *relative level* which is $2(\lambda_0 - \mu_0)$ if $\mathfrak{g} = F_4$ and $\lambda_0 - \mu_0$ otherwise. Although it is difficult to use (5.14) to compute $b_{\lambda, \mu}$ in general, one can nevertheless obtain the following result

$$\begin{aligned} b_{\lambda, \lambda} &= 1 && \text{if } \lambda = \lambda^i \in P_{\text{aty}}^1, i \geq 2 \text{ or } \lambda = \lambda_\pm^i \in P_{\text{aty}}^2, i \geq 1, \\ b_{\lambda, \lambda^{-1}} &= 1 && \text{if } \lambda = \lambda^2 \in P_{\text{aty}}^1, \\ b_{\lambda, \lambda^{-1}} &= 0 && \text{if } \lambda = \lambda^1 \in P_{\text{aty}}^1. \end{aligned} \quad (5.15)$$

[Some simplification takes place when $\mu = \lambda$ or $\mu = \lambda^{-1}$, as the relative level $|\lambda - \mu|$ is comparatively small. For instance in the first case it is controlled by (5.9).]

We now use this information and Lemma 5.5 to prove several claims, which will then imply the theorem.

Claim 1. *For any weight μ appearing in (5.10), which is either \mathfrak{g} -integral dominant or appears in the first graph, we have $a_{\lambda, \mu} = 1$.*

We already see $a_{\lambda, \mu} \leq b_{\lambda, \mu} = 1$. To prove $a_{\lambda, \mu} = 1$, first suppose $\lambda = \lambda^i$ or λ_\pm^i for $i \gg 0$. Then the only possible $\mu \prec \lambda$ is $\mu = \lambda$, which is λ^{i-1} or λ_\pm^{i-1} . As in the proof of (5.13), we see that there exists a \mathfrak{g} -highest weight vector with weight ν satisfying $\lambda - 2\rho_1 \preceq \nu \prec \lambda$ (this condition implies that ν must be \mathfrak{g} -integral dominant when $\lambda_0 \gg 0$), and by Lemma 5.5(2), λ is the only possible such μ . Thus $\lambda \in P(V(\lambda^{(i)}))$ for all $i \gg 0$.

Now assume conversely there exists some $i_0 > 0$ such that for $\lambda = \lambda^{i_0}$ or $\lambda_\pm^{i_0}$, $a_{\lambda, \mu} = 0$ for some said μ in the claim (i.e., either $\mu = \lambda$, or else $\mu = \lambda^{-1} \in P_{\text{aty}}^1$ with

$i_0 = 2$). By the previous paragraph, we can choose i_0 to be maximal. Then we have the following facts:

- $V(\lambda)$ contains a \mathfrak{g}_0 -highest weight μ by (5.15);
- any composition factor $L(\eta)$ of $V(\lambda)$ other than $L(\lambda)$ does not contain a \mathfrak{g}_0 -highest weight μ simply because either $\eta \prec \lambda = \mu$ or else $i_0 = 2$, $\lambda = \lambda^2 \in P_{\text{aty}}^1$, $\eta = \lambda^1$, $\mu = \lambda^{-1}$ (in this latter case $L(\eta) = L(\lambda^1)$ cannot contain a \mathfrak{g}_0 -highest weight $\mu = \lambda^{-1}$ as $V(\eta) = V(\lambda^1)$ does not contain one by (5.15)).

These facts imply that only the composition factor $L(\lambda)$ of $V(\lambda)$ contains a \mathfrak{g}_0 -highest weight μ . Since $L(\lambda)$ is a composition factor of $V(\lambda)$ by the maximal choice of i_0 , we deduce that $V(\lambda)$ contains a \mathfrak{g}_0 -highest weight μ . However $\mu \notin \Omega_\lambda$, a contradiction with (5.5). This proves the claim except the case that μ appears in the first graph of (5.10).

To see $a_{\lambda^0, \lambda_+^{-1}} = 1$, let us look at $V(\lambda_+^1)$, which contains the finite dimensional composition factor $L(\lambda^0)$ as we have just proved in the previous paragraph (note that $\lambda^0 = (\lambda_+^1)^\vee = (\lambda_-^1)^\vee$). Since the parabolic Verma module $V(\lambda_+^1)$ cannot contain a finite-dimensional submodule, we see that there must exist some weight $\nu \notin P^+$ such that

$$\lambda^0 \twoheadrightarrow \nu \text{ in the graph } P(V(\lambda_+^1)). \quad (5.16)$$

Such a ν must be contained in $P(V(\lambda_+^1))$ and in $P(V(\lambda^0))$, this is because a module $M(\lambda^0 \twoheadrightarrow \nu)$ (cf. Notation 5.2) with graph $\lambda^0 \twoheadrightarrow \nu$ must be a highest weight module with highest weight λ^0 , and hence a quotient of $V(\lambda^0)$. By Lemma 5.5(2), $\nu = \lambda_+^{-1}$ is the only possible weight. Thus $a_{\lambda^0, \lambda_+^{-1}} = 1$. The uniqueness of ν in (5.16) in fact also implies

$$\lambda^0 \rightarrow \lambda_+^{-1} \text{ in the graph } P(V(\lambda^0)), \quad (5.17)$$

i.e., this is the only possible connection between λ^0 and λ_+^{-1} in $P(V(\lambda^0))$. Similarly, considering $V(\lambda_-^1)$ instead of $V(\lambda_+^1)$ implies that we have $\lambda^0 \rightarrow \lambda_-^{-1}$ in $P(V(\lambda^0))$. This not only completes the proof of Claim 1 but also proves the first graph of (5.10).

Claim 2. *We have the second graph of (5.10).*

In this case, $\lambda = \lambda^1 \in P_{\text{aty}}^1$, and we have $\Omega_\lambda = \{\lambda^1, \lambda^{-1}, \lambda^{-2}\}$. However, $\lambda^{-1} \notin P(V(\lambda^1))$ by (5.15), so $P(V(\lambda^1)) \subset \{\lambda^1, \lambda^{-2}\}$ (thus equality must hold since $V(\lambda^1) \neq L(\lambda^1)$). Let $v'_{\lambda^{-2}} \in V(\lambda^1)$ be any primitive vector with weight λ^{-2} . As in (5.11), we have

$$v'_{\lambda^{-2}} = u_1 w_1 \text{ for some } u_1 \in U(\mathfrak{g}^-)U(\mathfrak{g}_1), \text{ where } w_1 = f_\varphi^{\bar{\lambda}_0+1} v_\lambda. \quad (5.18)$$

Note from (5.11) that w_1 has weight $\lambda^{-1} - 2\rho_1$, and so

$$u_1 \text{ has weight } 2\rho_1 - (\lambda^{-1} - \lambda^{-2}). \quad (5.19)$$

Also note that $\prod_{\alpha \in \Delta_1^+} e_\alpha w_1 = 0$ as otherwise it would be a \mathfrak{g} -highest weight vector with weight λ^{-1} (cf. (5.12)). Let $x \in U(\mathfrak{g}_1)$ be an element with maximal weight ξ such that $xw_1 \neq 0$ (so $\xi_1 := 2\rho_1 - \xi \succ 0$). By definition of x , the vector xw_1 is a \mathfrak{g} -highest weight vector with weight $\eta := \lambda^{-1} - \xi_1$ (so $\eta \in P(V(\lambda^1))$), thus the only possible η is $\eta = \lambda^{-2}$. Hence $\xi = 2\rho_1 - (\lambda^{-1} - \lambda^{-2})$, this together with (5.19) proves that u_1 and x has the same weight. Then the unique choice of x also shows that $u_1 \in U(\mathfrak{g}_1)$. From Definition 4.3, we see that $\lambda^{-1} - \lambda^{-2}$ can be uniquely written

as a sum of distinct roots in Δ_1^+ , accordingly, the element in $U(\mathfrak{g})$ with weight ξ is unique up to a nonzero scalar factor. This proves that $u_1 = x$ is unique up to a nonzero scalar factor (so $a_{\lambda^1, \lambda^{-2}} = 1$), and we have the second graph of (5.10).

From now on we shall consider the last two graphs of (5.10), i.e., $\lambda = \lambda^i \in P_{\text{aty}}^1$ with $i \geq 2$ or $\lambda = \lambda_{\pm}^i \in P_{\text{aty}}^2$ with $i \geq 1$.

Claim 3. *For any weight $\mu \notin P^+$ which does not appear in any of the last two graphs of (5.10), $a_{\lambda, \mu} = 0$.*

Note that the only possible weight in Ω_{λ} which does not appear in the graphs is the weight $\mu = (\lambda^{\sigma_0})^{\sim} = \lambda^{1-i}$ with $\lambda = \lambda^i \in P_{\text{aty}}^1$ and $i \geq 3$ or $\mu = (\lambda^{\sigma_0})^{\sim} = \lambda_{\pm}^{1-i}$ with $\lambda = \lambda_{\pm}^i \in P_{\text{aty}}^2$ and $i \geq 2$. In either case, $(\lambda^{\sigma_0})^{\sim} \notin P^+$. Assume $v'_{(\lambda^{\sigma_0})^{\sim}}$ is a primitive vector with weight $(\lambda^{\sigma_0})^{\sim}$. Then $v'_{(\lambda^{\sigma_0})^{\sim}} \in U(\mathfrak{g})f_{\varphi}^{\bar{\lambda}_0+1}v_{\lambda}$, but the maximal possible weight of $U(\mathfrak{g})f_{\varphi}^{\bar{\lambda}_0+1}v_{\lambda}$ is λ^{σ_0} (as we have seen in (5.11)), which is $\prec (\lambda^{\sigma_0})^{\sim}$, a contradiction. Thus $a_{\lambda, \mu} = 0$.

Claim 4. *If a weight μ appearing in any one of the last two graphs of (5.10) satisfies the conditions $\mu \neq \lambda$ and $\mu \in P^+$, then we have the subgraph $\lambda \rightarrow \mu$ in the graph $P(V(\lambda))$.*

We have $a_{\lambda, \mu} = 1$ by Claim 1. If there exists some weight ν with $\lambda \twoheadrightarrow \nu \twoheadrightarrow \mu$ in the graph $P(V(\lambda))$, then $\nu \notin P^+$ by Lemma 5.5(2) (otherwise we would have $\lambda = \lambda^2 \in P_{\text{aty}}^1$ and either $\nu = \lambda^1, \mu = \lambda^{-1}$ or else $\nu = \lambda^{-1}, \mu = \lambda^1$, but we already know there does not exist a graph $\lambda^1 \twoheadrightarrow \lambda^{-1}$ nor $\lambda^{-1} \twoheadrightarrow \lambda^1$ as a module $M(\lambda^1 \twoheadrightarrow \lambda^{-1})$ with the first graph would be a quotient of $V(\lambda^1)$ and the second would be some kind of dual of the first, namely, $M(\lambda^{-1} \twoheadrightarrow \lambda^1) = M(\lambda^1 \twoheadrightarrow \lambda^{-1})^{\vee}$). Therefore the primitive vector v'_{ν} with weight ν must be in $U(\mathfrak{g})f_{\varphi}^{\bar{\lambda}_0+1}v_{\lambda}$ and thus, so is the primitive vector v'_{μ} with weight μ . This means that $L(\mu)$ is not a composition factor of the Kac-module $K(\lambda)$, a contradiction with the maximality of $K(\lambda)$.

From now on we assume $\lambda = \lambda^i \in P_{\text{aty}}^1$ with $i \geq 2$ as the proof for the case $\lambda = \lambda_{\pm}^i \in P_{\text{aty}}^2$ with $i \geq 1$ is analogous.

Claim 5. *We have the subgraph $\lambda^{i-1} \rightarrow \lambda^{-i}$ in the graph $P(V(\lambda^i))$.*

Recall from (5.4) that $\lambda^{-i} = \lambda^{\sigma_0}$. First suppose $v'_{\lambda^{-i}}$ is a primitive vector with weight λ^{-i} , which must be in $U(\mathfrak{g})f_{\varphi}^{\bar{\lambda}_0+1}v_{\lambda}$. Thus as in the proof of (5.11) and (5.12), such $v'_{\lambda^{-i}}$ is unique, i.e., $a_{\lambda^i, \lambda^{-i}} \leq 1$. We already know $\lambda^{i-1} = \lambda^{\vee} \in P(V(\lambda))$, thus there must exist some weight $\nu \notin P^+$ such that

$$\lambda^{i-1} \twoheadrightarrow \nu \text{ in } P(V(\lambda^i)), \quad (5.20)$$

this is because $L(\lambda^{i-1})$ is finite dimensional and $V(\lambda^i)$ does not contain a finite dimensional submodule. As in the proof of (5.17), such a weight ν is in $P(V(\lambda^i)) \cap P(V(\lambda^{i-1}))$ and thus must be λ^{-i} (thus $a_{\lambda^i, \lambda^{-i}} \geq 1$), and furthermore, the uniqueness of ν in (5.20) also implies

$$\lambda^{i-1} \rightarrow \lambda^{-i} \text{ in the graph } P(V(\lambda^i)), \text{ and no } \eta \text{ with } \lambda^{-i} \twoheadrightarrow \eta. \quad (5.21)$$

Claim 6. *We have the subgraph $\lambda^i \rightarrow \lambda^{-i-1} \rightarrow \lambda^{-i}$ in the graph $P(V(\lambda^i))$.*

Loot at the graph $P(V(\lambda^{i+1}))$, by Claim 5 or (5.21), we have a subgraph $\lambda^i \rightarrow \lambda^{-i-1}$. Since a module $M(\lambda^i \rightarrow \lambda^{-i-1})$ with graph $\lambda^i \rightarrow \lambda^{-i-1}$ must be a quotient of $V(\lambda^i)$, we obtain that $\lambda^{-i-1} \in P(V(\lambda^i))$ and $\lambda^i \rightarrow \lambda^{-i-1}$ is a subgraph of $P(V(\lambda^i))$. Now assume $v'_{\lambda^{-i-1}}$ is a primitive vector with weight λ^{-i-1} . We want to prove $v'_{\lambda^{-i-1}}$ is unique.

Let M_1 be the submodule of $V(\lambda^i)$ generated by the primitive vector with weight λ^{-i} , and set $M = V(\lambda^i)/M_1$. Note from (5.21) that M_1 does not have a primitive weight λ^{-i-1} (in fact M_1 is the simple module $L_{\lambda^{-i}}$ by (5.21)), thus $v'_{\lambda^{-i-1}}$ uniquely corresponds to a primitive vector (also denoted by the same symbol) in $P(M)$. Now as in the proof of Claim 2, such a primitive vector in $P(M)$ is unique, and it must have the form $v'_{\lambda^{-i-1}} = u_1 w_1$ for some $u_1 \in U(\mathfrak{g}_1)$ (cf. (5.18)), i.e., $u_1 = \prod_{\alpha \in S} e_\alpha$ for some subset S of Δ_1^+ . Now return to the parabolic Verma module $V(\lambda^i)$, we obtain that a primitive vector with weight λ^{-i-1} is unique, which is $v'_{\lambda^{-i-1}} = \prod_{\alpha \in S} e_\alpha w_1$. Since the primitive vector with weight λ^{-i} is $v'_{\lambda^{-i}} = \prod_{\alpha \in \Delta_1^+} e_\alpha w_1$ (as in the proof of Claim 5), which can be then written as $v'_{\lambda^{-i}} = u_2 v'_{\lambda^{-i-1}}$ with $u_2 = \prod_{\alpha \in \Delta_1^+ \setminus S} e_\alpha$, i.e., $\lambda^{-i-1} \twoheadrightarrow \lambda^{-i}$ in $P(V(\lambda^i))$. As there is no possible primitive weight sitting in between λ^{-i-1} and λ^{-i} , we have $\lambda^{-i-1} \rightarrow \lambda^{-i}$.

We have completed the proof of Theorem 5.6. \square

6. PROOFS OF MAIN THEOREMS ON JANTZEN FILTRATION

6.1. Rigidity of parabolic Verma modules. Note that one can read the radical filtration of a module in \mathcal{O}^p off its primitive weight graph. To see this, let $P(V)$ be the multi-set of primitive weights of a module $V \in \mathcal{O}^p$, which will be regarded as a set in the way explained in Remark 5.3. We decompose the set into a disjoint union of subsets $P(V) = \cup_{i \geq 0} P(V)_i$ in the following way. The subset $P(V)_0$ consists of the primitive weights which are not derived from any weights. A primitive weight μ belongs to $P(V)_i$ if in the skeleton of $P(V)$, the longest of the oriented paths from weights in $P(V)_0$ to μ has i arrows. Let $V^{(i)}$ be the submodule of V generated by all primitive vectors in $\cup_{j \geq i} P(V)_j$. Then we obtain the following filtration for V ,

$$V = V^{(0)} \supset V^{(1)} \supset \dots \supset V^{(\ell)} \supset V^{(\ell+1)} = 0, \quad (6.1)$$

where the *length* ℓ of the filtration is the smallest non-negative integer such that $P(V)_{\ell+1} = \emptyset$. We will say that elements of $P(V)_i$ are at level i .

Lemma 6.1. *The filtration (6.1) is the radical filtration of V . Furthermore, the consecutive quotients of the filtration are given by*

$$V_i := V^{(i)}/V^{(i+1)} = \bigoplus_{\lambda \in P(V)_i} L(\lambda), \quad i = 0, 1, \dots, \ell.$$

Proof. This is obvious from the definition of a primitive weight graph. \square

Example 6.2. Consider as an example a module V with the skeleton of its primitive weight graph given by Figure 1.

The radical filtration of V has length 2, with the non-empty $P(V)_i$ given by

$$P(V)_0 = \lambda, \quad P(V)_1 = \{\mu, \mu', \mu''\}, \quad P(V)_2 = \nu,$$

and the consecutive quotients of the radical filtration given by

$$V_0 = L(\lambda), \quad V_1 = L(\mu) \oplus L(\mu') \oplus L(\mu''), \quad V_2 = L(\nu).$$

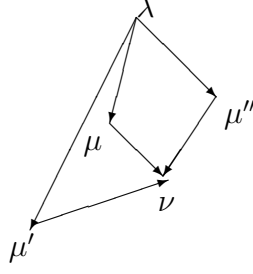


FIGURE 1. Example

The radical filtration is the unique Loewy filtration in this case, as one can immediately see by looking at the graph.

Now we can easily prove Theorem 3.2.

Proof of Theorem 3.2. We can construct the radical filtration of $V(\lambda)$ by applying Lemma 6.1 to its primitive weight graph, which is one of the graphs obtained in Theorem 5.6. By inspecting the graph, we immediately see that the radical filtration of $V(\lambda)$ is the unique Loewy filtration. Note that the third graph in (5.10) is a special case of Figure 1, which was already considered in detail in Example 6.2. \square

6.2. Proof of Theorem 3.3. The claim of Theorem 3.3 is trivially true if $V(\lambda)$ is simple, thus we assume that $V(\lambda)$ is reducible. We only need to show that the Jantzen filtration is Loewy since $V(\lambda)$ is rigid by Theorem 3.2. From Theorem 5.6 we can see that $V(\lambda)$ is multiplicity free, namely, all composition factors have multiplicity one. It then follows from the proof of [32, Theorem 3.6] that the Jantzen filtration has semi-simple consecutive quotients.

It remains to prove that the length ℓ (cf. (6.1)) of the Jantzen filtration is minimal, i.e., $\ell = 1$ in all cases except that $\ell = 2$ if λ is in the cases of the last two graphs of (5.10).

First we suppose that λ is a \mathfrak{g} -integral and \mathfrak{g}_0 -dominant atypical weight such that $\lambda = \lambda^i, \lambda_{\pm}^i$ with $i \geq 2$ or $\lambda = \lambda_{\pm}^1 \in P_{\text{aty}}^2$ (i.e., λ is in the last two cases of (5.10)). We assume $\lambda = \lambda^i$ as the proof for the case $\lambda = \lambda_{\pm}^i$ is similar. We only need to prove that the socle $L(\lambda^{-i})$ of $V(\lambda^i)$ is contained in $V^2(\lambda)$.

Let $v'_{\lambda^{-i}}$ be a primitive vector with weight λ^{-i} . Then it is a nonzero highest weight vector of $L(\lambda^{-i}) \subset V(\lambda^i)$. Up to a nonzero scalar factor, $v'_{\lambda^{-i}}$ is equal to $\prod_{\alpha \in \Delta_1^+} e_{\alpha} w_1$ with $w_1 = f_{\varphi}^{\bar{\lambda}_0+1} v_{\lambda}$ as in the proof of Theorem 5.6. The product of e_{α} can be ordered so that $\prod_{\alpha \in \Delta_1^+} e_{\alpha} = \prod_{\alpha \in \Delta_1^{\pm}} e_{\alpha} \prod_{\alpha \in \Delta_1^{\mp}} e_{\alpha}$. Then up to a nonzero scalar factor,

$$v'_{\lambda^{-i}} = \prod_{\alpha \in \Delta_1^{\pm}} e_{\alpha} \cdot \prod_{\alpha \in \Delta_1^{\mp}} f_{\alpha} \cdot \tilde{f} f_{\varphi}^{\bar{\lambda}_0+1-r_1} v_{\lambda}, \quad (6.2)$$

where $r_1 = \#\Delta_1^{\mp}$, and $\tilde{f} = f_{\delta}$ if $\mathfrak{g} = G_3$ or $\tilde{f} = 1$ else. This can be shown by noting that $[e_{\alpha}, f_{\beta}] = 0$ for $\alpha \in \Delta_1^{\mp}, \beta \in \Delta_1^{\pm}$, and $[e_{\alpha}, f_{\varphi}] = f_{\bar{\alpha}}$ (up to a nonzero scalar factor) for $\alpha \in \Delta_1^{\mp}$, where $\bar{\alpha} = w_0(\alpha)$ with $w_0 \in W_0$ being the product of the all σ_i 's in (2.15). It is the unique root in Δ_1^{\pm} obtained from α by changing all signs of ε_i with $i > 0$ (cf. (2.2)–(2.4)).

We use the same symbols $v'_{\lambda^{-i}}$ and v_{λ} to denote the corresponding vectors in the deformed parabolic Verma module $V_T(\lambda^i)$. Let $D := \langle v'_{\lambda^{-i}}, v'_{\lambda^{-i}} \rangle$, which is a

polynomial in t . Since every nonzero submodule of $V(\lambda^i)$ must contain the highest weight vector of the socle $L(\lambda^{-i})$ of $V(\lambda^i)$, we conclude that the lowest order term of the polynomial D is of a degree equal to the length ℓ of the Jantzen filtration, that is, $D = t^\ell u(t)$ for some polynomial $u(t)$ in t with a nonzero constant term.

For convenience, we define $\tilde{e} = e_\delta$ if $\mathfrak{g} = G_3$ and else $\tilde{e} = 1$. Upon using (6.2) (but interpreting $v'_{\lambda^{-i}}$ as in $V_T(\lambda^i)$), we immediately see from (3.11) that

$$Dv_\lambda = e_\varphi^{\bar{\lambda}_0+1-r_1} \tilde{e} \prod_{\alpha \in \Delta_1^\pm} e_\alpha \prod_{\alpha \in \Delta_1^\pm} f_\alpha v'_{\lambda^{-i}}. \quad (6.3)$$

Let $v'' = \prod_{\alpha \in \Delta_1^\pm} e_\alpha \prod_{\alpha \in \Delta_1^\pm} f_\alpha v'_{\lambda^{-i}}$. Observe from (6.2) that $e_\alpha v'_{\lambda^{-i}} = 0$ for all $\alpha \in \Delta_1^\pm$. Similar computations as those in (5.13) show that up to a nonzero factor in \mathbb{C} ,

$$v'' = p_1(t) v'_{\lambda^{-i}} \quad \text{with} \quad p_1(t) := \prod_{\alpha \in \Delta_1^\pm} (\lambda_{(t)}^{-i} + \rho, \alpha), \quad (6.4)$$

where we have adopted the notation that $\mu_{(t)} = \mu + t\delta$ for any $\mu \in \mathfrak{h}^*$. Thus

$$Dv_\lambda = p_1(t) e_\varphi^{\bar{\lambda}_0+1-r_1} \tilde{e} v'_{\lambda^{-i}}.$$

The proof of (6.4) is done by case by case computations for all the exceptional Lie superalgebras. Consider as an example the case with $\mathfrak{g} = D(2, 1; a)$. Then up to nonzero factors in \mathbb{C} ,

$$\begin{aligned} v'' &= e_{\delta-\varepsilon_1-\varepsilon_2} e_{\delta-\varepsilon_1+\varepsilon_2} f_{\delta-\varepsilon_1+\varepsilon_2} f_{\delta-\varepsilon_1-\varepsilon_2} v'_{\lambda^{-i}} \\ &= e_{\delta-\varepsilon_1-\varepsilon_2} [e_{\delta-\varepsilon_1+\varepsilon_2}, f_{\delta-\varepsilon_1+\varepsilon_2}] f_{\delta-\varepsilon_1-\varepsilon_2} v'_{\lambda^{-i}} \\ &= (\delta - \varepsilon_1 + \varepsilon_2, \lambda_{(t)}^{-i} - (\delta - \varepsilon_1 - \varepsilon_2)) e_{\delta-\varepsilon_1-\varepsilon_2} f_{\delta-\varepsilon_1-\varepsilon_2} v'_{\lambda^{-i}} \\ &= (\delta - \varepsilon_1 + \varepsilon_2, \lambda_{(t)}^{-i} + \rho) [e_{\delta-\varepsilon_1-\varepsilon_2}, f_{\delta-\varepsilon_1-\varepsilon_2}] v'_{\lambda^{-i}} \\ &= (\delta - \varepsilon_1 + \varepsilon_2, \lambda_{(t)}^{-i} + \rho) (\delta - \varepsilon_1 - \varepsilon_2, \lambda_{(t)}^{-i}) v'_{\lambda^{-i}} \\ &= (\delta - \varepsilon_1 + \varepsilon_2, \lambda_{(t)}^{-i} + \rho) (\delta - \varepsilon_1 - \varepsilon_2, \lambda_{(t)}^{-i} + \rho) v'_{\lambda^{-i}} \\ &= \prod_{\alpha \in \Delta_1^\pm} (\alpha, \lambda_{(t)}^{-i} + \rho) v'_{\lambda^{-i}} = p_1(t) v'_{\lambda^{-i}}, \end{aligned}$$

where the third and fifth equalities follow respectively from that $(\delta - \varepsilon_1 + \varepsilon_2, -\delta + \varepsilon_1 + \varepsilon_2) = (\delta - \varepsilon_1 + \varepsilon_2, \rho)$ and $(\delta - \varepsilon_1 - \varepsilon_2, \rho) = 0$ (cf. (2.13)).

Now we return to an arbitrary exceptional Lie superalgebra. Since $\lambda^{-i} = \lambda^{\sigma_0}$, we have $\lambda_{(t)}^{-i} + \rho = \sigma_0(\lambda^i + \rho) + t\delta = \sigma_0(\lambda^i + \rho - t\delta)$. Hence

$$p_1(t) = \prod_{\alpha \in \Delta_1^\pm} (\sigma_0(\lambda^i + \rho - t\delta), \alpha) = tp_{12}(t), \quad (6.5)$$

with $p_{12}(t)$ being a polynomial in t with a nonzero constant term, where the last equality follows by noting the following facts. For $\alpha \in \Delta_1^\pm$, $(\sigma_0(\lambda^i + \rho - t\delta), \alpha) = (\lambda^i + \rho - t\delta, \sigma_0(\alpha)) = (\lambda^i + \rho - t\delta, -\bar{\alpha})$, where $\bar{\alpha}$ is defined immediately after (6.2). There exists a unique atypical root γ of λ such that $\gamma \in \Delta_1^\pm$ since λ is non-tail (cf. (4.14) and statements after it).

Next let us compute $e_\varphi^{\bar{\lambda}_0+1-r_1} \tilde{e} v'_{\lambda-i}$. Up to a nonzero factor in \mathbb{C} ,

$$e_\varphi^{\bar{\lambda}_0+1-r_1} \tilde{e} v'_{\lambda-i} = \tilde{e} \prod_{\alpha \in \Delta_1^\pm} e_\alpha e_\varphi^{\bar{\lambda}_0+1-r_1} f_\varphi^{\bar{\lambda}_0+1-r_1} w, \quad (6.6)$$

where $w := \prod_{\alpha \in \Delta_1^\pm} f_\alpha \tilde{f} v_\lambda$, with “weight” $\mu(t) := \lambda_{(t)}^i - r_1 \varsigma \delta - 2\delta_{\mathfrak{g}, G_3} \delta$ (where $\varsigma = \frac{1}{2}$ if $\mathfrak{g} = F_4$ or 1 else, $\delta_{\mathfrak{g}, G_3} = 1$ if $\mathfrak{g} = G_3$ or 0 else). Note that $e_\varphi w = 0$. Thus the factor $e_\varphi^{\bar{\lambda}_0+1-r_1} f_\varphi^{\bar{\lambda}_0+1-r_1}$ on the right-hand side of (6.6) can be easily eliminated, leading to (up to a nonzero factor in \mathbb{C})

$$t(\lambda_0 + 1 - r_1)! \prod_{k=1}^{\lambda_0-r_1} (t+k) w', \quad (6.7)$$

where $w' = \prod_{\alpha \in \Delta_1^\pm} e_\alpha \tilde{e} f \prod_{\alpha \in \Delta_1^\pm} f_\alpha v_\lambda$. Calculations similar to those leading to (6.4) and (5.13) reduce w' to

$$w' = p_{22}(t) v_\lambda \quad \text{with} \quad p_{22}(t) = p_{21}(t) \prod_{\alpha \in \Psi} (\alpha, \lambda_t + \rho), \quad (6.8)$$

where $p_{21}(t) = 1$ and $\Psi = \Delta_1^\pm$ if $\mathfrak{g} \neq G_3$, or else, $p_{21}(t) = \lambda_0 + t - r_1$ (up to a nonzero factor in \mathbb{C}) and $\Psi = \Delta_1^\pm \setminus \{\delta\}$. For proving (6.8), one observes in the latter case that up to nonzero scalar factors, $\tilde{e} = e_\delta$, $e_\delta^2 = e_{2\delta}$, $f_\delta^2 = f_{2\delta}$ and $[e_{2\delta}, f_{2\delta}] = h_\varphi$; the factors $e_\delta, \tilde{e}, \tilde{f}, f_\delta$ in w' are arranged to appear in this order. Note that in the case $\mathfrak{g} = G_3$, $P_{\text{aty}}^2 = \emptyset$, and for $\lambda = \lambda^i \in P_{\text{aty}}^1$ with $i \geq 2$ we have $\lambda_0 > 3 = r_1$ by (4.13). Thus $p_{22}(t)$ is a polynomial with a nonzero constant term.

The long computation finally gives

$$D = \langle v'_{\lambda-i}, v'_{\lambda-i} \rangle = t^2 p_{12}(t) p_{22}(t) (\lambda_0 + 1 - r_1)! \prod_{k=1}^{\lambda_0-r_1} (t+k),$$

and $t^{-2}D$ is a polynomial with a nonzero constant term. This shows that the Jantzen filtration in this case is of length 2. By Lemma 6.1, this is the Loewy length. Hence the Jantzen filtration is a Loewy filtration, which is unique by Theorem 3.2.

Next suppose λ is in the first case of (5.10), i.e., $\lambda = \lambda^0 \in P_{\text{aty}}^2$. Take $v_+ = \prod_{\alpha \in \Delta_1^\pm} f_\alpha v_\lambda$ and $v_- = \prod_{\alpha \in \Delta_1^\pm} f_\alpha v_\lambda$. Then $\prod_{\alpha \in \Delta_1^\pm} e_\alpha v_+ = 0 = \prod_{\alpha \in \Delta_1^\pm} e_\alpha v_-$ as in the proof of (5.13). Arguments similar to those following (5.13) show that there exist some $u_\pm \in U(\mathfrak{g}_{-1})$ such that $v'_{\lambda_\pm^{-1}} = u_\pm v_\lambda$ are primitive vectors with weights λ_\pm^{-1} respectively. Then the same methods used in the earlier part of this proof show that $v'_{\lambda_\pm^{-1}} \in V^1(\lambda)$, i.e., the Jantzen filtration has length 1, and hence is the unique Loewy filtration.

The other case can be proven in a similar way but much more simply as the primitive weight graph is simpler. We omit the details.

6.3. Computation of u^- -homology groups. In this section, we compute the homology groups $H_i(u^-, L(\lambda))$. The results will be needed for proving Theorem 3.5. We remark that the results are interesting in their own right.

Theorem 6.3. *Let λ be an atypical \mathfrak{g} -integral weight.*

(1) If $\lambda = \lambda^i \in P_{\text{aty}}^1$ for some $i \in \mathbb{Z}^*$, then as \mathfrak{g}_0 -modules,

$$H_k(\mathfrak{u}^-, L(\lambda^i)) \cong \begin{cases} L^0(\lambda^{i-k}) & \text{if } k = 0 \text{ or } i \leq -1, \\ L^0(\lambda^{-k-1}) & \text{if } k \geq i = 1, \\ L^0(\lambda^{-i-k}) \oplus L^0(\lambda^{-k+i-2}) & \text{if } k \geq i \geq 2, \\ L^0(\lambda^{-i-k}) \oplus L^0(\lambda^1) \oplus L^0(\lambda^{-1}) & \text{if } k = i - 1 \geq 1, \\ L^0(\lambda^{-i-k}) \oplus L^0(\lambda^{i-k}) & \text{if } 1 \leq k \leq i - 2. \end{cases} \quad (6.9)$$

(2) If $\lambda = \lambda_+^i \in P_{\text{aty}}^2$ for some $i \in \mathbb{Z}$, then as \mathfrak{g}_0 -modules,

$$H_k(\mathfrak{u}^-, L(\lambda_+^i)) \cong \begin{cases} L^0(\lambda_+^{i-k}) & \text{if } k = 0 \text{ or } i \leq -1, \\ L^0(\lambda_+^{-k}) \oplus L^0(\lambda_-^{-k}) & \text{if } k > i = 0, \\ L^0(\lambda_-^{i-k}) \oplus L^0(\lambda_+^{-i-k}) & \text{if } k \geq i \geq 1, \\ L^0(\lambda_+^{i-k}) \oplus L^0(\lambda_+^{-i-k}) & \text{if } 1 \leq k < i. \end{cases} \quad (6.10)$$

Proof. We will prove (1) only as (2) can be proven similarly. For any \mathfrak{u}^- -module V , we denote $H_k(V) := H_k(\mathfrak{u}^-, V)$ for simplicity. Consider $V(\lambda^{-i})$ for $i \geq 1$. Part (2) of Theorem 5.6 gives the short exact sequence

$$0 \rightarrow L(\lambda^{-i-1}) \rightarrow V(\lambda^{-i}) \rightarrow L(\lambda^{-i}) \rightarrow 0,$$

from which arises the following long exact sequence of homology groups:

$$\begin{aligned} \cdots \rightarrow H_k(L(\lambda^{-i-1})) \rightarrow H_k(V(\lambda^{-i})) \rightarrow H_k(L(\lambda^{-i})) \rightarrow \\ \rightarrow H_{k-1}(L(\lambda^{-i-1})) \rightarrow H_{k-1}(V(\lambda^{-i})) \rightarrow H_{k-1}(L(\lambda^{-i})) \rightarrow \cdots \end{aligned} \quad (6.11)$$

Since the parabolic Verma module $V(\lambda)$ for any λ is a free \mathfrak{u}^- -module, we always have (hereafter the “equality” always means the “ \mathfrak{g}_0 -module isomorphism”)

$$H_k(V(\lambda)) = \begin{cases} L^0(\lambda) & \text{if } k = 0, \\ 0 & \text{otherwise,} \end{cases} \quad (6.12)$$

where $H_0(V(\lambda))$ is obtained from its definition. In fact,

$$H_0(V(\lambda)) = V(\lambda)/\mathfrak{u}^-V(\lambda) = L^0(\lambda) = L(\lambda)/\mathfrak{u}^-L(\lambda) = H_0(L(\lambda)).$$

Thus (6.11) gives

$$H_k(L(\lambda^{-i})) = L^0(\lambda^{-i-k}) \text{ for } k \geq 0, i \geq 1. \quad (6.13)$$

Similarly, from the short exact sequence $0 \rightarrow L(\lambda^{-2}) \rightarrow V(\lambda^1) \rightarrow L(\lambda^1) \rightarrow 0$ (cf. the second graph of (5.10)), we obtain

$$H_k(L(\lambda^1)) = \begin{cases} L^0(\lambda^1) & \text{if } k = 0, \\ L^0(\lambda^{-k-1}) & \text{otherwise.} \end{cases} \quad (6.14)$$

Now consider $V(\lambda^i)$ with $i \geq 2$. We let M_i, M_1 be respectively the \mathfrak{g} -modules with primitive weight graphs

$$M_i : \lambda^{-i-1} \leftarrow \lambda^i \text{ and } M_1 : \lambda^{-1} \rightarrow \lambda^{-2} \leftarrow \lambda^1. \quad (6.15)$$

Then the third and fourth graphs of (5.10) show that we have the short exact sequence

$$0 \rightarrow M_{i-1} \rightarrow V(\lambda^i) \rightarrow M_i \rightarrow 0. \quad (6.16)$$

Note that the subgraph of M_1 obtained by deleting λ^{-1} is the primitive weight graph for the parabolic Verma module $V(\lambda^1)$ (cf. the second graph of (5.10)). Therefore, we have the short exact sequence

$$0 \rightarrow V(\lambda^1) \rightarrow M_1 \rightarrow L(\lambda^{-1}) \rightarrow 0,$$

which gives rise to a long exact sequence of homology groups. Since the homology groups of both $L(\lambda^{-1})$ and $V(\lambda^1)$ are all known by (6.14) and (6.12), and in particular, $H_k(V(\lambda^1)) = 0$ for all $k > 0$, this long exact sequence determines

$$H_k(M_1) = \begin{cases} L^0(\lambda^1) \oplus L^0(\lambda^{-1}) & \text{if } k = 0, \\ L^0(\lambda^{-k-1}) & \text{otherwise.} \end{cases} \quad (6.17)$$

Analogously, from (6.16) and (6.12), we obtain for $i \geq 2$,

$$H_k(M_i) = H_{k-1}(M_{i-1}) = \begin{cases} L^0(\lambda^{-k+i-2}) & \text{if } k \geq i \geq 2, \\ L^0(\lambda^1) \oplus L^0(\lambda^{-1}) & \text{if } k = i-1 \geq 1, \\ L^0(\lambda^{i-k}) & \text{if } 0 \leq k \leq i-2. \end{cases} \quad (6.18)$$

The primitive weight graph of M_i for $i \geq 2$ in (6.15) yields the following short exact sequence: $0 \rightarrow L(\lambda^{-i-1}) \rightarrow M_i \rightarrow L(\lambda^i) \rightarrow 0$. Thus we have the long exact sequence

$$\begin{aligned} \cdots \rightarrow H_k(L(\lambda^{-i-1})) &\xrightarrow{\psi_k} H_k(M_i) \rightarrow H_k(L(\lambda^i)) \rightarrow \\ &\rightarrow H_{k-1}(L(\lambda^{-i-1})) \xrightarrow{\psi_{k-1}} H_{k-1}(M_i) \rightarrow H_{k-1}(L(\lambda^i)) \rightarrow \cdots \end{aligned} \quad (6.19)$$

Note that all maps in (6.19) are \mathfrak{g}_0 -module homomorphisms. By inspecting (6.18), we immediately see that as a \mathfrak{g}_0 -module, $H_k(M_i)$ does not have a composition factor $L^0(\lambda^{-i-k-1})$. Since $H_k(L(\lambda^{-i-1})) = L^0(\lambda^{-i-k-1})$ by (6.13), the \mathfrak{g}_0 -module homomorphism ψ_k must be zero. Hence

$$\begin{aligned} H_k(L(\lambda^i)) &= H_{k-1}(L(\lambda^{-i-1})) \oplus H_k(M_i) \\ &= \begin{cases} L^0(\lambda^{-i-k}) \oplus L^0(\lambda^{-k+i-2}) & \text{if } k \geq i \geq 2, \\ L^0(\lambda^{-i-k}) \oplus L^0(\lambda^1) \oplus L^0(\lambda^{-1}) & \text{if } k = i-1 \geq 1, \\ L^0(\lambda^{-i-k}) \oplus L^0(\lambda^{i-k}) & \text{if } 1 \leq k \leq i-2 \\ L^0(\lambda^i) & \text{if } 0 = k \leq i-2. \end{cases} \end{aligned} \quad (6.20)$$

From this together with (6.13) and (6.14), we obtain (6.9). \square

6.4. Proof of Theorem 3.5. Theorem 3.5 is equivalent to

$$\sum_{\mu \in P_0^+} J_{\lambda\mu}(q) p_{\mu\nu}(q) = \delta_{\lambda\nu} \quad \text{for all } \lambda, \nu \in P_0^+. \quad (6.21)$$

Since the consecutive quotients $V(\lambda)_i$ of the Jantzen filtration are semisimple,

$$\sum_{\mu} [V(\lambda)_i : L(\mu)] [H_j(L(\mu)) : L^0(\nu)] = [H_j(V(\lambda)_i) : L^0(\nu)].$$

Thus, the left-hand side of (6.21) can be expressed as

$$\begin{aligned} & \sum_{\mu, i, j} q^{i+j} (-1)^j [V(\lambda)_i : L(\mu)] [H_j(L(\mu)) : L^0(\nu)] \\ &= \sum_k q^k \sum_{j=0}^k (-1)^j [H_j(V(\lambda)_{k-j}) : L^0(\nu)]. \end{aligned}$$

Note that the constant term of the right-hand side of this equation is obviously equal to $\delta_{\lambda\nu}$. Thus the proof of (6.21) is equivalent to showing

$$\sum_{j=0}^k (-1)^j H_j(V(\lambda)_{k-j}) = 0 \text{ for } k \geq 1, \quad (6.22)$$

where the left-hand side is interpreted as an element in the Grothendieck group of the category $\mathcal{O}_{\mathfrak{g}_0}$ of $U(\mathfrak{g}_0)$ -modules.

We shall only consider in detail the case of an atypical $\lambda \in P^+$ such that either $\lambda = \lambda^i \in P_{\text{aty}}^1$ for some $i \geq 2$ or $\lambda = \lambda_{\pm}^i \in P_{\text{aty}}^2$ with $i \geq 1$, as in all the other cases the parabolic Verma module $V(\lambda)$ has much simpler structure by Theorem 5.6, and the proof of (6.22) is considerably easier. Now assume $\lambda = \lambda^i \in P_{\text{aty}}^1$ with $i \geq 2$. Since the Jantzen filtration (3.13) coincides with the radical filtration (6.1), we obtain from (5.10) that

$$V(\lambda^i)_0 = L(\lambda^i), \quad V(\lambda^i)_1 = L(\lambda^{i-1}) \oplus L(\lambda^{-i-1}) \oplus \delta_{i2} L(\lambda^{-1}), \quad V(\lambda^i)_2 = L(\lambda^{-i}),$$

and $V(\lambda^i)_k = 0$ for $k > 2$. Using the result (6.9) on \mathfrak{u}^- -homology groups, one immediately obtains (6.22). The proof is similar if $\lambda = \lambda_{\pm}^i \in P_{\text{aty}}^2$ with $i \geq 1$.

7. CHARACTERS, DIMENSIONS AND COHOMOLOGY GROUPS OF FINITE DIMENSIONAL SIMPLE MODULES

7.1. Character and dimension formulae for simple modules. Theorem 5.6 enables us to derive a character formula and dimension formula for the atypical finite-dimensional simple modules in a way analogous to the proofs of [33, Theorem 4.4] and [31, Theorem 4.16]. For an atypical weight $\lambda \in P^+$, we define

$$S_{\lambda} = \{\lambda, \lambda^{\sigma_0}\} \cap \{\nu \in P^+ \mid \nu \preccurlyeq \lambda\}, \quad m_{\lambda} = \#(\{\lambda, \lambda^{\sigma_0}\} \cap \{\nu \in P^+ \mid \nu \succcurlyeq \lambda\}),$$

where λ^{σ_0} is defined in (2.16). Then it is easy to see that

$$S_{\lambda} = \begin{cases} \{\lambda, \lambda^{\sigma_0}\} & \text{if } \lambda \succ \lambda^{\sigma_0} \in P^+, \\ \{\lambda\} & \text{otherwise,} \end{cases} \quad m_{\lambda} = \begin{cases} 2 & \text{if } \lambda \prec \lambda^{\sigma_0}, \\ 1 & \text{otherwise.} \end{cases}$$

For $\mu \in S_{\lambda}$, we denote $\theta_{\lambda, \mu} \in W$ to be the unique element with minimal length such that $\theta_{\lambda, \mu} \cdot \lambda = \mu$, namely, $\theta_{\lambda, \mu} = 1$ if $\lambda = \mu$ or $\theta_{\lambda, \mu} = \sigma_0$ otherwise. Using the same method as that for the proof of [33, Theorem 4.4], we obtain from Theorem 5.6 the following result.

Theorem 7.1. *Let \mathfrak{g} be an exceptional Lie superalgebra, and let $L(\lambda)$ be the finite-dimensional irreducible \mathfrak{g} -module with atypical highest weight λ .*

(1) *The character of $L(\lambda)$ is given by*

$$\text{ch } L(\lambda) = \sum_{\mu \in S_{\lambda}} \frac{(-1)^{|\theta_{\lambda, \mu}|}}{m_{\mu} R_{\bar{0}}} \sum_{w \in W} \text{sign}(w) w \left(e^{\mu + \rho_{\bar{0}}} \prod_{\beta \in \Delta_1^+ \setminus \{\gamma_{\mu}\}} (1 + e^{-\beta}) \right), \quad (7.1)$$

where $R_{\bar{0}}$ is defined by (3.1), γ_μ is the atypical root of $\mu \in S_\lambda$, and $|\theta_{\lambda,\mu}|$ is the length of $\theta_{\lambda,\mu}$.

(2) The dimension of $L(\lambda)$ is given by

$$\dim L(\lambda) = \sum_{\mu \in S_\lambda, B \subset \Delta_1^+ \setminus \{\gamma_\mu\}} (-1)^{|\theta_{\lambda,\mu}|} m_\mu^{-1} \prod_{\alpha \in \Delta_0^+} \frac{(\alpha, \rho_{\bar{0}} + \mu - \sum_{\beta \in B} \beta)}{(\alpha, \rho_{\bar{0}})}. \quad (7.2)$$

(3) The character of the finite-dimensional (typical or atypical) Kac \mathfrak{g} -module $K(\lambda)$ is $\text{ch } K(\lambda) = \chi^V(\lambda)$ with the right-hand side given by (3.6) unless $\lambda^{\sigma_0} \in P^+$. If $\lambda^{\sigma_0} \in P^+$, then $\text{ch } K(\lambda) = \text{ch } L(\lambda)$ with the right-hand side given by (7.1).

7.2. First and second cohomology groups. Applying the methods used in the proofs of [33, Theorem 5.1] and [30, Theorems 1.1–1.3] to the present case, we obtain the following result from the structure theorem (Theorem 5.6) of parabolic Verma modules.

Theorem 7.2. *Let \mathfrak{g} be an exceptional Lie superalgebra. Let $L(\lambda)$ and $K(\lambda)$ respectively denote the finite-dimensional irreducible and Kac modules with highest weight λ . Let $\Lambda^i \in P_{\text{aty}}^1$, $i = -1$ or $i \geq 1$ be defined by $\Lambda^{-1} = 0$, $\Lambda^1 = (\Lambda^{-1})^\vee$, $\Lambda^i = (\Lambda^{i-1})^\vee$ for $i \geq 2$. Then*

$$\begin{aligned} H^1(\mathfrak{g}, L(\lambda)) &\cong \begin{cases} \mathbb{C} & \text{if } \lambda = \Lambda^2, \\ 0 & \text{otherwise.} \end{cases} \\ H^1(\mathfrak{g}, K(\lambda)) &\cong \begin{cases} \mathbb{C} & \text{if } \lambda = \Lambda^3, \\ 0 & \text{otherwise.} \end{cases} \\ H^2(\mathfrak{g}, L(\lambda)) &\cong \begin{cases} \mathbb{C} & \text{if } \lambda = \Lambda^1, \Lambda^3, \\ 0 & \text{otherwise.} \end{cases} \\ H^2(\mathfrak{g}, K(\lambda)) &\cong \begin{cases} \mathbb{C} & \text{if } \lambda = \Lambda^1, \Lambda^4. \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

8. COMMENTS ON JANTZEN FILTRATION FOR ORTHOSYMPLECTIC LIE SUPERALGEBRAS

In this final section, we briefly comment on Jantzen filtration of parabolic Verma modules over the remaining basic classical simple Lie superalgebras, the orthosymplectic Lie superalgebras $\mathfrak{osp}_{m|2n}$ with $m \neq 2$ ($\mathfrak{osp}_{2|2n}$ is type I). Again we fix the distinguished maximal parabolic subalgebra \mathfrak{p} of $\mathfrak{osp}_{m|2n}$ and consider the parabolic category $\mathcal{O}^{\mathfrak{p}}$ of \mathbb{Z}_2 -graded $\mathfrak{osp}_{m|2n}$ -modules.

For $n = 1$, we can easily establish properties analogous to Theorems 3.3 and 3.5 by using results in [33]. This was already alluded to in [32].

Theorem 8.1. *The Jantzen filtration of the parabolic Verma module $V(\lambda)$ over $\mathfrak{osp}_{m|2}$ is the unique Loewy filtration. Furthermore, for any $\lambda, \mu \in P_0^+$, the Jantzen polynomials $J_{\lambda\mu}(q)$ defined in (3.16) coincide with the inverse Kazhdan-Lusztig polynomials $a_{\lambda\mu}(q)$.*

Proof. Theorem 4.2 in [33] is the precise analogue of Theorem 5.6 for $\mathfrak{osp}_{k|2}$ with a slight change of notation. Since the arguments in §6 depend only on the primitive

weight graphs in Theorem 5.6, they all go through in the present case, leading to the theorem. We omit the details. \square

Finally for $\mathfrak{osp}_{m|2n}$ with $m \neq 2$ and $n > 1$, super duality [10] for orthosymplectic Lie superalgebras will enable one to relate aspects of the Jantzen filtration for parabolic Verma modules over $\mathfrak{osp}_{m|2n}$ to those of the Jantzen filtration for parabolic Verma modules over orthogonal Lie algebras. Even though the Jantzen filtration for the latter is not well understood except for the cases corresponding to Hermitian symmetric pairs [13, 4], this nevertheless leads to useful insights into the problem at hand. We will treat the Jantzen filtration for the orthosymplectic Lie superalgebras in a future publication.

Acknowledgement: This work was supported by the Australian Research Council (grant no. DP0986551), the National Science Foundation of China (grant no. 10825101), the Shanghai Municipal Science and Technology Commission (grant no. 12XD1405000) and the Fundamental Research Funds for the Central Universities of China.

REFERENCES

- [1] H.H. Andersen, *Filtrations of cohomology modules for Chevalley groups*, Ann. Sci. École Norm. Sup. (4) **16** (1983), no. 4, 495–528
- [2] H.H. Andersen, *Jantzen’s filtrations of Weyl modules*, Math. Z. **194** (1987), no. 1, 127–142.
- [3] A.A. Beilinson and J. Bernstein, *Localisation de g -modules*, C. R. Acad. Sci. Paris Ser. I Math. **292** (1981), no. 1, 15–18.
- [4] A.A. Beilinson and J. Bernstein, *A proof of Jantzen conjectures*, I. M. Gelfand Seminar, 1–50, Adv. Soviet Math., **16**, Part 1, Amer. Math. Soc., Providence, RI, 1993.
- [5] B.D. Boe and D. H. Collingwood, *Multiplicity free categories of highest weight representations. I, II*, Comm. Algebra **18** (1990), no. 4, 947–1032, 1033–1070.
- [6] J. Brundan, *Kazhdan-Lusztig polynomials and character formulae for the Lie superalgebra $\mathfrak{gl}(m|n)$* , J. Amer. Math. Soc **16** (2003), 185–231.
- [7] J. Brundan and C. Stroppel, *Highest weight categories arising from Khovanov’s diagram algebra IV: the general linear supergroup*, J. Eur. Math. Soc. (JEMS) **14** (2012), no. 2, 373–419.
- [8] J.-L. Brylinski and M. Kashiwara, *Kazhdan-Lusztig conjecture and holonomic systems*, Invent. Math. **64** (1981), no. 3, 387–410.
- [9] S.J. Cheng and N. Lam, *Irreducible characters of general linear superalgebra and super duality*, Comm. Math. Phys. **298** (2010), 645–672.
- [10] S.J. Cheng, N. Lam and W. Wang, *Super duality and irreducible characters of orthosymplectic Lie superalgebras*, Invent. Math. **183** (2011), 189–224.
- [11] S.J. Cheng, W. Wang and R.B. Zhang, *Super duality and Kazhdan-Lusztig polynomials*, Trans. American Math. Soc. **360** (2008), 5883–5924.
- [12] S.J. Cheng and R.B. Zhang, *Analogue of Kostant’s u -cohomology formula for the general linear superalgebra*, International Math. Research Notices (2004), no. 1, 31–53.
- [13] D.H. Collingwood, R.S. Irving and B. Shelton, *Filtrations on generalized Verma modules for Hermitian symmetric pairs*, J. Reine Angew. Math. **383** (1988), 54–86.
- [14] P. Fiebig, *Centers and translation functors for the category \mathcal{O} over Kac-Moody algebras*, Math. Z. **243** (2003), no. 4, 689–717.
- [15] O. Gabber and A. Joseph, *Towards the Kazhdan-Lusztig conjecture*, Ann. Sci. École Norm. Sup. (4) **14** (1981), no. 3, 261–302.
- [16] J.E. Humphreys, *Representations of semisimple Lie algebras in the BGG category \mathcal{O}* , Graduate Studies in Mathematics, **94**, American Mathematical Society, Providence, RI, 2008, xvi+289 pp.

- [17] R.S. Irving, *A filtered category \mathcal{O}_S and applications* (and *List of Errata*), Mem. Amer. Math. Soc. **83** (1990), no. 419, vi+117 pp.
- [18] J.C. Jantzen, *Kontravariante Formen auf induzierten Darstellungen halbeinfacher Lie-Algebren*, Math. Ann. **226** (1977), no. 1, 53–65.
- [19] J.C. Jantzen, *Moduln mit einem höchsten Gewicht*, Lecture Notes in Mathematics, **750**, Springer, Berlin, 1979.
- [20] V.G. Kac, *Lie superalgebras*, Adv. Math. **26** (1977), 8–96.
- [21] V.G. Kac, *Characters of typical representations of classical Lie superalgebras*, Comm. Alg. **5** (1977), 889–897.
- [22] V.G. Kac, *Representations of classical Lie superalgebras*, Lect. Notes Math. **676** (1978), 597–626.
- [23] V.G. Kac, M. Wakimoto, *Integrable highest weight modules over affine superalgebras and number theory*, in Lie Theory and Geometry, 415–456, Progress in Math., 123, Birkhauser Boston, Boston, MA, 1994.
- [24] D. Kazhdan and G. Lusztig, *Representations of Coxeter groups and Hecke algebras*, Invent. Math. **53** (1979), no. 2, 165–184.
- [25] M. Scheunert, *The theory of Lie superalgebras. An introduction*, Lecture Notes in Mathematics, **716**, Springer, Berlin, 1979.
- [26] V. Serganova, *Kazhdan-Lusztig polynomials and character formula for the Lie superalgebra $gl(m|n)$* , Selecta Math. **2** (1996), 607–654.
- [27] W. Soergel, *Andersen filtration and hard Lefschetz*, Geom. Funct. Anal. **17** (2008), no. 6, 2066–2089.
- [28] C. Stroppel, *Parabolic category \mathcal{O} , perverse sheaves on Grassmannians, Springer fibres and Khovanov homology*, Compositio Math. **145** (2009), 954–992.
- [29] Y. Su, J.W.B. Hughes and R.C. King, *Primitive vectors in the Kac-module of the Lie superalgebra $sl(m|n)$* , J. Math. Phys. **41** (2000), 5044–5087.
- [30] Y. Su and R.B. Zhang, *Cohomology of Lie superalgebras $\mathfrak{sl}_{m|n}$ and $\mathfrak{osp}_{2|2n}$* , Proc. London Math. Soc. **94** (2007), 91–136.
- [31] Y. Su and R.B. Zhang, *Character and dimension formulae for general linear superalgebra*, Adv. Math. **211** (2007), 1–33.
- [32] Y. Su and R.B. Zhang, *Generalised Jantzen filtration of Lie superalgebras I*, J. Eur. Math. Soc. **14** (2012), 1103–1133.
- [33] Y. Su and R.B. Zhang, *Generalised Verma modules for the orthosymplectic Lie superalgebra $\mathfrak{osp}_{k|2}$* , Journal of Algebra **357** (2012), 94–115.
- [34] J. Van der Jeugt, *Irreducible representations of the exceptional Lie superalgebras $D(2, 1; \alpha)$* , J. Math. Phys. **26** (1985), 913–924.
- [35] J. Van der Jeugt and R.B. Zhang, *Characters and composition factor multiplicities for the Lie superalgebra $gl(m|n)$* , Lett. Math. Physics, **47** (1999), 49–61.

DEPARTMENT OF MATHEMATICS, TONGJI UNIVERSITY, SHANGHAI 200092, CHINA
E-mail address: ycsu@tongji.edu.cn

SCHOOL OF MATHEMATICS AND STATISTICS, UNIVERSITY OF SYDNEY, SYDNEY, AUSTRALIA
E-mail address: ruibin.zhang@sydney.edu.au